January 2015

₹30

MATHEMATICS Spectrum

90 DAYS REVISION Planner for JEE

JEE FINAL LAP

Quick Review of Concepts

Complex Number
Quadratic Equations, Inequalities
Sequence and Series

FROM CLASS 11 SYLLABUS

RELATION
FUNCTION
CONTINUITY AND DIFFERENTIABILITY

FROM CLASS 12 SYLLABUS

TEST RIDER

Test Series for JEE Main & Advanced

JEE Main Test Ride 1
JEE Main Test Ride 2
JEE Advanced Test Ride 1

REGULARS

Master the NCERT 11 & 12
JEE Advanced Corner
Brain Teasers
Concept Map etc

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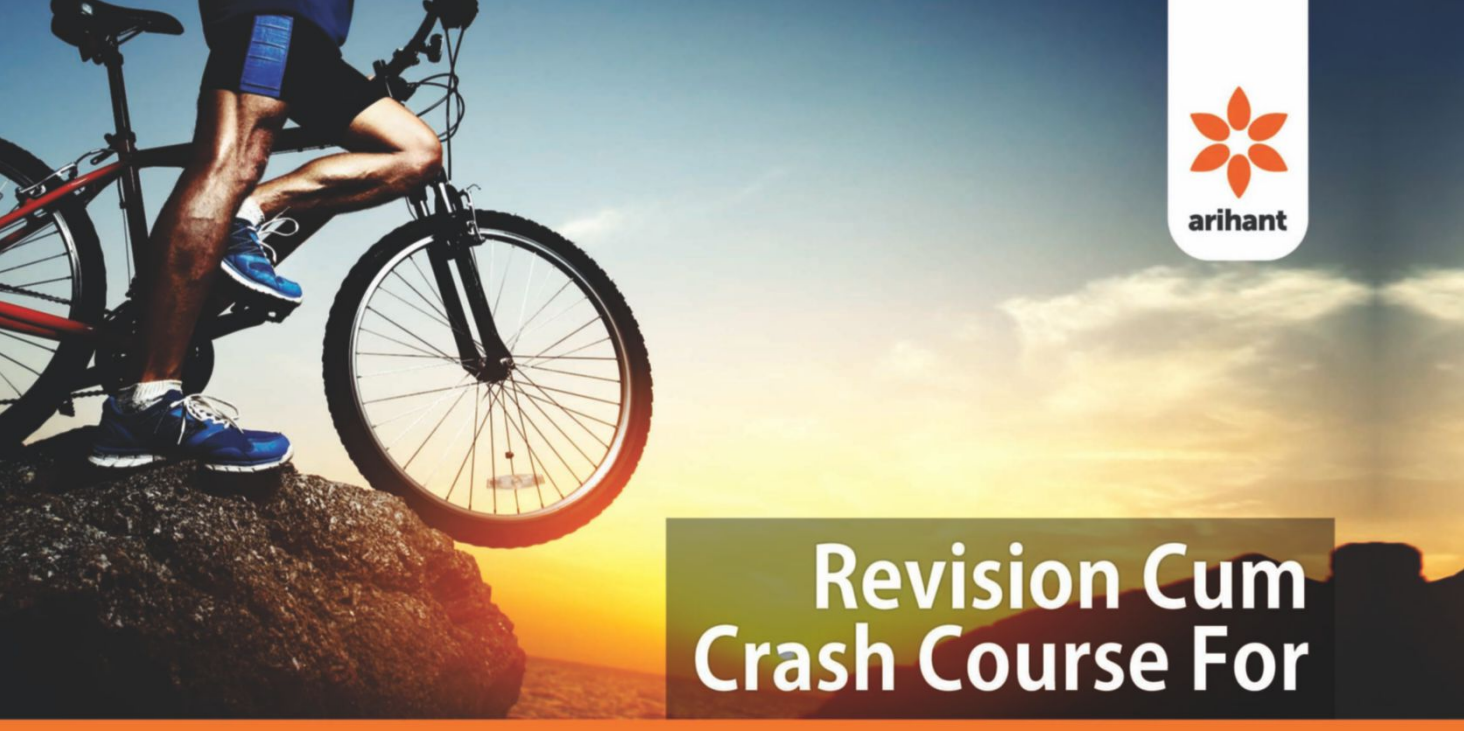

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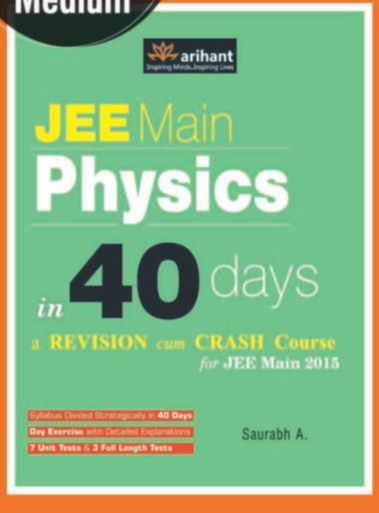

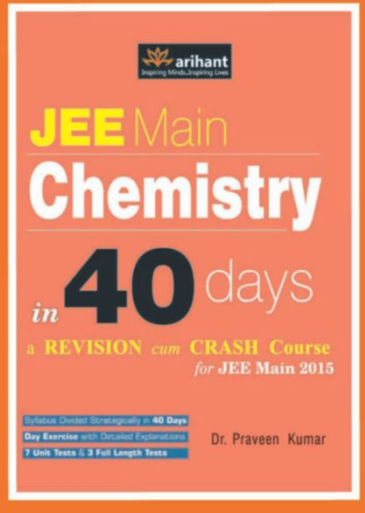

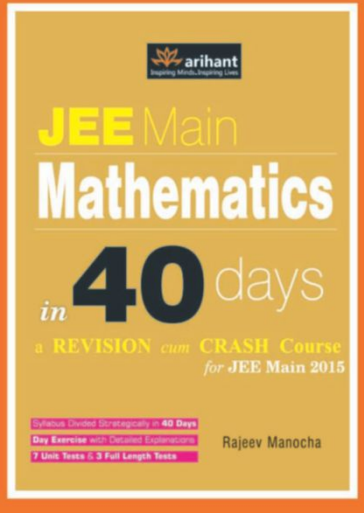

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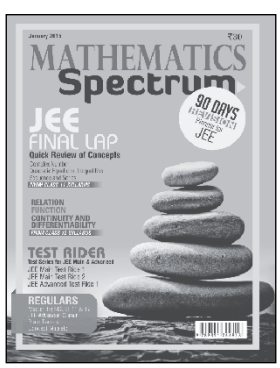
Issue 3 **Vol 1**



Dear Aspirants,

I like January, because it is the month in which, I can see a whole year stretching out ahead of me. Slowly, 2014 is winding down, and 2015 is knocking at our doors. A New Year always brings new hopes, new plans, new strategies, and new resolutions to our life.

In the last issue I told you that we are planning something special for you. As the 90 days countdown for your exams begin, we came up with certain new features to make your task achievable. These features include:

- A "90 days countdown" to plan up the studies of next 90 days in the subject
- "Quick concept review" is designed to strengthen up your learning in a faster way for the 'JEE final lap'.
- "Test Rider" is a test series of total 14 tests (9 for mains and rest for advanced) to be covered up in next 4 issues
- "Test Drive" is another test series of 14 tests (5@class 11th, and 9@class 12th) to be covered up in next 3 issues.
- A brush up article with revision tips covering do's and don'ts related to 'Revision: the art to deliver'.

This last lap of 90 days is very crucial for any aspirant and from the side of spectrum family I personally assure you that we will try to create more and more to help you out. Along with all these one personal piece of advice from my side is "Try to manage your time as you manage your Money". The spectrum team will be anxiously waiting for your feed backs and comments.

GOOD LUCK

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90 DAYS COUNTDOWN
towards JEE 2015

1. Set, Relation & Function

DAY 1
- Sets, Venn Diagram Algebra

DAY 2
- Types of Relation, Equivalence Relation

DAY 3
- One-one, Into and Onto Function, Inverse Function

DAY 4
- Composition of Function and Binary Operations

2. Matrix and Determinants

DAY 5
- Matrices, Algebra of Matrices, Transpose of Matrix

DAY 6
- Symmetric & Skew-symmetric Matrices, Transformation of a Matrix, Inverse of Matrix

DAY 7
- Determinants & Properties of Determinants, Minor and Cofactors

DAY 8
- Adjoint of Matrix, Solution of Linear Equations by Cramer's Rule

3. Trigonometry (9 to 14 Days)

DAY 9
- Trigonometric Basics, Trigonometric Ratios, Ratios of Compound Angles

DAY 10
- Maximum-Minimum Values and Trigonometric Periodic Functions

DAY 11
- Trigonometric Equations, Transformation Formulae

DAY 12
- Inverse Trigonometric Functions, Inverse Hyperbolic Functions

DAY 13
- Solutions of Triangle, Radii of Inscribed & Circumscribed Circles of Regular Polygon

DAY 14
- Heights and Distances

EXPERTS SPEAK Topic 1, day 1 is important from board point of view. Topic 2 day 1, venn diagram is very helpful in solving practical problems based on sets,however, this portion of syllabus is very helpful to understand the concept of probability,as well.In JEE usually 1 question is asked from topics 1 & 2 of day 4.Topic 4 day 8 is useful in solving practical problems based on application of matrix. Topics 1 & 2 day 12 are very important for JEE specially graph of Trigonometric and inverse trigonometric functions for solving questions in very short span of time.Topics 2 & 3 day 13 and topic 1 day 14 is also important specially for finding incentre and excenter of in circle and escribed circle and in finding the height of tower or building in different situations.

As most of the part of first 14 days serves as the basis for gripping the subject, thus, it is advisable not to move on day 15th topics until you develop confidence in these topics.

4. Calculus (15 to 35 Days)

DAY 15
- Limits

DAY 16
- Continuity and Differentiability

DAY 17
- Differentiation Formulae, Differentiation by Substitution

DAY 18
- Higher Order Derivatives and Partial Derivative

DAY 19
- Mean Value Theorems, Maxima-Minima of Functions of One Variable

DAY 20
- Maxima-Minima & Tangents & Normals

DAY 21
- Indefinite Integrals Fundamental

DAY 22
- Integration by Substitution Parts & Partial Functions

DAY 23
- Fundamental of Definite Integral

DAY 24
- Definite Integration by Substitution Parts and Partial Functions

DAY 25
- Properties of Summation of Series of Definite Integration

DAY 26
- Definite Integration by Gamma Function, Leibnitz Rule, Walli's Formulae

DAY 27
- Area of Curves

DAY 28
- Volume and Surface Area of Solids

DAY 29
- Concept of Ordinary Differential Equations and their Order and Degree

DAY 30
- Formation of Differential Equations and Solution of Differential Equations

DAY 31
- Homogeneous & Linear Differential Equations Application and Standard Results of Solutions

DAY 32
- Revision of Calculus

DAY 33
- Test of Calculus

EXPERTS SPEAK Topic 1 day 15 & day 16 are important for JEE as generally 1 question is asked every year,however, this portion of syllabus series as the basic need to grip the differential calculus. Topic 1 day 19 is important and has wide application such as in finding the roots of a polynomial within the given interval. Topics 1 and 2 day 20 are important from board as well as JEE point of view. Aspirants are advised to focus on these topics.

Topics 1 and 2 of day 22 are very helpful in gripping the basics of integral calculus. The topics related to definite integral of days 24, 25 & 26 are very important for JEE and has several application based questions such as in finding the sum of a given series, while Leibnitz rule are useful for differentiation under integrationsign.Topics based on application of definite integration are very important for direct questions,while all topics of day 31 are important for direct as well as application based questions.Random and planned testing is required to judge the exact learning. For topics underlined above learning with direct solved examples will be beneficial.

5. Mathematical Induction

DAY 34
- Mathematical Induction and its Applications

6. Binomial Theorem

DAY 35
- Binomial Theorem and its Simple Applications

DAY 36
- Properties of Binomial Coefficients and Multinomial Theorem

DAY 37
- Practice Test of Binomial & Mathematical Induction

7. Permutations & Combinations

DAY 38
- Permutation & Simple Applications

DAY 39
- Combination as Selection

DAY 40
- Practice of Permutation & Combination

8. Sequence & Series

DAY 41
- Concept of AP,GP & HP

DAY 42
- Relations between AP,GP & HP, Relation between AM and GM

DAY 43
- Sum of Special Series

DAY 44
- Logarithmic & Exponential Series

DAY 45
- Practice of Sequence & Series

EXPERTS SPEAK Topic 34 is less important from JEE point of view but important for divisibility based problems. Topics of binomial theorem, till day 37, are important and has several application based problems.It also helps in solving the problems based on combinations. Topics on day 38 and 39 are important as they have wide application and are useful to solve problems of probability. Topic after day 45th belongs to class XIth and need to recall more than once. All the topics till day 42nd have problems related to *n*th,sum to *n* terms and also sum of infinite term are important to be asked as direct and application based questions. All the topics till day 44 are required essentially for basics of logarithm and exponential functions. Students are advised to focus on some special series like arithmetico-geometric series and telescopicsum. Use random testing for topics till day 45th for self analysis and use planned tests in two section i.e.

(a) Day 1st to 23rd (b) Day 24th to 45th for proper assessment and to gain confidence.

9. Coordinate Geometry

DAY 46
- Cartesian System of Rectangular Coordinates

DAY 47
- Straight Line

DAY 48
- Circle & its Equation, Tangent & Normal to a Circle

DAY 49
- Chord of Contact of Tangent Pole & Polar System of Circles

DAY 50
- Parabola & its Equation to Tangent

DAY 51
- Director Circle, Pole & Polar, Length of Tangent & Normal

DAY 52
- The Ellipse

DAY 53
- Hyperbola, Chord , Tangent

DAY 54
- Asymptote , Conjugates, Rectangular Hyperbola

DAY 55
- Practice Test of Coordinate Geometry

DAY 56
- Coordinate of a Point in Space, Direction Ratios and Direction Cosines , Angle between Two Intersecting Lines

DAY 57
- Skew-lines, Shortest Distance between Them & its Equation

DAY 58
- Equation of a Line & Plane in Different Forms, Intersection of a Line & Plane, Coplanar Lines

10. Vector Algebra

DAY 59
- Vector & Scalars, Algebra of Vectors in 2D & 3D

DAY 60
- Scalar & Vector Products, Scalar & Vector Triple Products

DAY 61
- Test of 3D & Vector Algebra

11. Statistics & Probability

DAY 62
- Mean, Median & Mode of Grouped & Ungrouped Data

DAY 63
- Standard Deviation, Variance & Mean Deviation for Grouped & Ungrouped Data

DAY 64
- Practice Test of Statistics

DAY 65
- Probability of an Event Addition & Multiplication Theorem

DAY 66
- Baye's Theorem & Probability Distribution

DAY 67
- Bernoulli's Trial & Binomial Distribution

DAY 68
- Practice Test of Probability

EXPERTS SPEAK All the topics till day 58th are very important as 1 or 2 questions are asked in JEE specially director circle, pole and polar, length of tangent normal related to different conics. In three dimensional geometry, skew-lines, shortest distance between them, Intersections of lines and planes, coplanar, lines. Students are advised to focus on these topics. Topics 1 and 2 of day 60 are important and have direct and application based questions. All the topics till day 61 have wide application in mathematics as well as in physics. Topic 2 day 65, topic 1 day 66 and topic 1 day 67 are very important as every year 1 question is asked in JEE from them of direct as well as application based nature.Begin this complete section everyday with randomized check through random test to analysis your points of emphasis for your preparation.

12. Linear Programming

DAY 69
- Linear Programming Introduction, Objective Function, Optimization

DAY 70
- Types of LPP, Mathematical & Graphical Solution for Problems in Two Variables

DAY 71
- Feasibles & Infeasible Region & Optimal Feasible Solutions

DAY 72
- Practice Test

13. Complex Numbers & Quadratic Equations

DAY 73
- Complex Numbers Representation in a Plane Argand Diagram

DAY 74
- Algebra of Complex Numbers, Modulus & Argument

DAY 75
- Square Root & Triangle Inequality of Complex Numbers

DAY 76
- Quadratic Equation in Real & Complex Numbers System & their Solution

DAY 77
- Relation between Roots & Coefficients

DAY 78
- Nature of Roots & Formation of Quadratic Equations with Given Roots

DAY 79
- Practice of Complex Numbers & Quadratic Equations

DAY 80
- Test of Algebra

14. Mathematical Reasoning

DAY 81
- Mathematical Reasoning Logic & Truth Table

DAY 82
- Tautology, Contradiction, Converse & Contrapositive Quantifiers

DAY 83
- Unit Test of Calculus

DAY 84
- Unit Test of Trigonometry

DAY 85
- Unit Test of Coordinate

DAY 86
- Unit Test of Vector & 3D Geometry

DAY 87
- Unit Test of Probability & Mathematical Reasoning

DAY 88
- Practice Set (Full Syllabus)

DAY 89
- Practice Set (Full Syllabus)

DAY 90
- Revision of Important Points

EXPERTS SPEAK This last segment of your preparation is important as considerable number of questions is asked in JEE generally from the topics properties of modulus and argument triangle inequality rotation of complex numbers.

All the topics till day 79 have direct and application based questions. Topic of day 81 and 82 is important from JEE Mains point of view as every year 1 questions are asked. Concepts of these topics are useful in boolean algebra as well as in computer organisation.

Try to complete the schedule of last 20 days in 17 days (approx) as some loose days (days with lesser syllabus) are given in between. In the last 3 days use randomized testing of complete syllabus to filter your weaker portion and utilize these days to plug your weak points left.

For gaining the confidence after wards use 2-3 simulator papers of full syllabus and try to complete them approximately 15 min before the stipulated time, so that you can achieve the same under examination conditions of stress within the stipulated time.

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Revision The Art to Deliver

Human mind is very complicated and have an unlimited capacity to learn, which is almost similar in all of us. Inspite of such a huge capacity most of us deliver or perform only up to average or near about levels. Only a very few show 'out of box' or 'extra ordinary' performances. Terms like **success, toppers** etc., are associated with these very few persons only. Do you know, what makes these different? What they have which most of us do not have? When we look in to the answers of these questions, one sentence strikes always in my mind i.e. "Toppers are in the habit of doing the things differently." Here, surely the word **differently** has the art of revision or reconciliation of learning in its basis especially for the toppers of entrance examinations.

Why Revision

Revision plays a prominent role in preparation of school, college or other entrance exams. Once syllabus is covered, revision is a mandatory requirement for better scoring, even, if one had not covered 100% of syllabus but only 70% or so, still he has to revise what syllabus has been covered.

It may even be useless to have covered vast syllabus and going for test or exams without proper revision. Because, all that vast topics one has covered might have erased out of memory or less remembered, thus require reconciliation to deliver properly.

Further more, Revision has certain qualities like

- The more number of times you revise the better is your confidence and scoring chances.
- You get further clarity, thought, understanding and also interest in the subject or topic you study by revision.
- Revision enhances memory of the subject in terms of formulas, structures (Biology, Chemistry), diagrams (in Biology, Physics, Geography etc.), steps of derivations (in Physics, Maths), topic etc.
- It gives you command over the topics which are complex, vaguely or incompletely known to you. Your standards may rise to the extent that you may be able to derive the formula you forgot just before answering.
- Revision increases the rate of problem solving i.e. your ability to answer the number of problems of Physics, Maths, Chemistry etc., in limited amount of time. This helps you buy time to answer more questions in the given time and also think of ways to solve unanswerable questions.
- If you think all the attributes mentioned above due to revision are required, then one might not ask why revision is necessary.

For Maths and Stats Revision

Then best way as everyone says, practice-practice-practice. But some more tips to consider

- Formulas; make a note of them on a plain paper and fix them before your bed or sort so as to view them frequently and memorise them. Because if you remember the formulas possibly the problem solving is half done.
- In mathematical problems you need to understand the flow of steps towards the answer but not mug-up the steps. Because, if you forget even a single step you may stop solving it. If you try to flow steps towards answer you can find one way on your own and also remember the step better.
- Try even answering new problems from question banks, past papers etc.
- Mathematical logics are developed with more and more exposure, so try to utilize more and more of your study time in the development of logics. This ensures higher success rate.
- This not only builds your confidence but also trains your mind to solve the problems faster.

Concepts-misconcepts @ revision

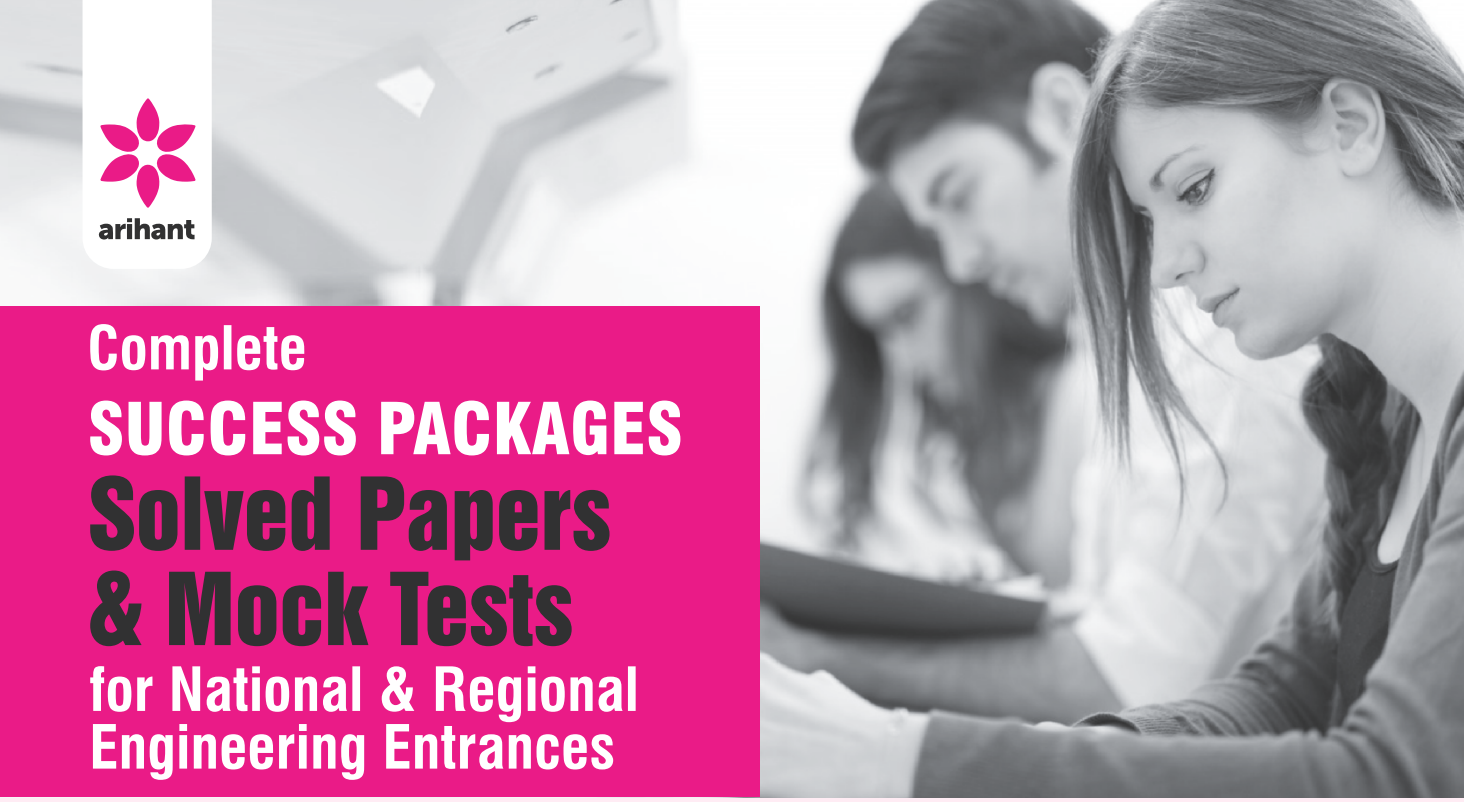
Remember, revision is not something that starts in the end of academic session rather it is what you do daily. It starts very first day when you learn something new. Everything that is taught in your class needs to be revised on the very same day. Many a times, we tend to miss out on writing some points when the teacher speaks. It is best to pen them down when they are still fresh in the memory.

Techniques to Revise Effectively

While going for revisions keep following points in mind

- It is always better to prepare a revision schedule on daily basis and after making the schedule, it is must to possess the will to abide by that. It is not necessary to revise large number of chaptersevery day.
- Revision must be slow and steady. It must be in the order to get firm grip on all types of questions asked on that topic.
- Revise as many times as possible the entire subjects.
- There is a huge difference between revision and repetition i.e. revision is not going through all the notes over and over again.
- The specialized feature of revision is lessening of time in each and every attempt i.e. time consumed in revising the same content every time is less than the previous one. A potential aspirant can revise the complete content of Physics syllabus in about two and a half hours in its 4th or 5th revision cycle.
- Decrease in the time of revision is possible only when you have a proper analysis of 'what to revise' and 'what not to revise'. Such an analysis can be done on the basis of 'what you know' and 'what is whipped off from your memory after learning'. Every time when you go for only those parts which are whipped off, you find a lesser content to revise. This, in turn, certainly requires lesser time to reconcile.
- Understand the concept and don't just mug. This is a big mistake made by many school going students till the mind maturity. Understanding the concept shows you the actual sweetness of the subject and makes you perceive even more of the subject with enthusiasm.
  The difference between mugging and understanding is that, mugging is a memorising aspect while understanding is realsing the facts about the subject or topic with its related prior knowledge.
- Once you revise try to answer a previous question paper i.e. past papers to see your progress.
- Some subjects may not be liked by us but try to accept that they are to be mastered for good scoring.
- Use same notes or books or study material for all the revisions because the page of the book also strikes in memory during answering the questions asked.
- There is no substitute for hard preparation and excitement and enthusiasm in revision.

Last but not least you are the best to know your situation, analyze it thoroughly and work out the best method for your revision. Think for yourself, you are mature enough to organize your own study plan. Use your discretion to judge what will work for you and what will not.



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JEE FINAL LAP
QUICK CONCEPT REVIEW
@ CLASS XI SYLLABUS

Complex Numbers

A number of the form $x + iy$, where $x, y \in R$ and $i = \sqrt{-1}$, is called a complex number. If $z = x + iy$, then x is called real part of z and denoted as Re(z) whereas y is called imaginary part of z and denoted as Im(z).

Note Complex number may be defined as an ordered pair of real numbers written as (x, y). A complex number is purely real, if $\text{Im}(z) = 0$ and it is purely imaginary, if $\text{Re}(z) = 0$.

Key Concept

- $(a + ib)(< \text{or} >)c + id$ is not defined i.e. order relation in complex number is not defined (comparison is only possible, if imaginary part is zero).
- $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is valid, if atleast one of a and b is non-negative. If a and b both are negative, then $\sqrt{a} \times \sqrt{b} = -\sqrt{ab}$.

Integral Power of Iota (i)

As we know square of every square number is positive, hence solution of the equation $x^2 = -9$ is not possible in real number system. $i = \sqrt{-1}$ is known as iota which is introduced in complex number system.

Also, $i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$

$$i^{4n} = 1,\ i^{4n+1} = i,\ i^{4n+2} = -1 \text{ and } i^{4n+3} = -i$$

SPEED BOOSTER The sum of four consecutive power of 'i' is zero i.e. $i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0$.

Conjugate of Complex Number

Complex numbers $z = a + ib$ and $\bar{z} = a - ib$ are called conjugate to each other (here, complex conjugate are obtained by just changing the sign of i).

Also, $\text{Re}(z) = \dfrac{z + \bar{z}}{2} = a$, $\quad (z) = \dfrac{z - \bar{z}}{2i} = b$

Properties of Conjugate of a Complex Number

If z is a complex number, then following are the properties of conjugate

(i) $\overline{(\bar{z})} = z$

(ii) $z + \bar{z} = 2\,\text{Re}(z)$, $z - \bar{z} = 2\text{Im}(z)$

(iii) $z + \bar{z} = 0 \Leftrightarrow$ z is purely imaginary; $z - \bar{z} = 0 \Leftrightarrow$ z is purely real.

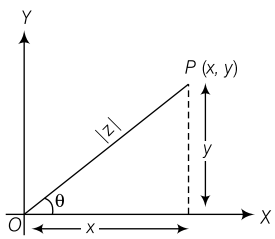
(iv) $z\bar{z} = [\text{Re}(z)]^2 + [\text{Im}(z)]^2$

(v) $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$; $\overline{(z_1 - z_2)} = \bar{z}_1 - \bar{z}_2$

(vi) $\overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2$, $\left(\dfrac{\overline{z_1}}{z_2}\right) = \dfrac{\bar{z}_1}{\bar{z}_2}$

Modulus and Argument of Complex Number

If complex number $z = x + iy$, then modulus of complex number z is $|z|$, which is distance of point 'z' from the origin and inclination of OP with positive X-axis is called argument of the complex number. Hence,



$$|z| = \sqrt{x^2 + y^2} \text{ and } \arg(z) = \tan^{-1}\left|\frac{y}{x}\right| + 2k\pi,\ k \in I$$

Note z is a point, whereas $|z|$ is distance of this point from origin.

Properties of Modulus and Argument

(i) $|z| = 0 \Leftrightarrow z = 0$; $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

(ii) $-|z| \le \text{Re}(z) \le |z|$; $-|z| \le \text{Im}(z) \le |z|$

(iii) $|z_1 z_2| = |z_1||z_2|$; $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, if $z_2 \neq 0$

(iv) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + \bar{z}_1 z_2 + z_1 \bar{z}_2$
$= |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1 \bar{z}_2)$

(v) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2\text{Re}(z_1\bar{z}_2)$

(vi) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

(vii) $\arg(\bar{z}) = -\arg(z)$

(viii) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

(ix) $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

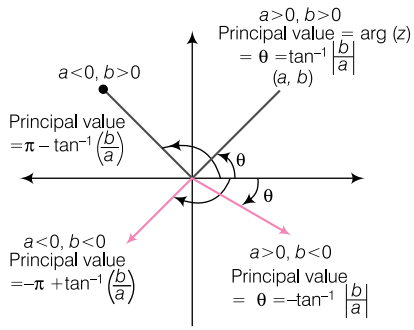
SPEED BOOSTER

(i) If $|z| = a$, then maximum and minimum values of $|z + c + id|$ is $a + \sqrt{c^2 + d^2}$ and $\left|a - \sqrt{c^2 + d^2}\right|$.

(ii) If $\left|z + \frac{1}{z}\right| = a$, then largest and smallest values of $|z|$ are $\frac{a + \sqrt{a^2 + 4}}{2}$ and $\frac{-a + \sqrt{a^2 + 4}}{2}$.

Principal Value of Argument

If $z = a + ib$, $a, b \in R$, then $\arg(z) = \tan^{-1}\left(\frac{b}{a}\right)$ always gives the principal value. It depends on quadrant in which the point (a, b) lies.

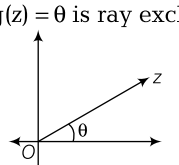


Note (i) $-\pi < \theta \le \pi$, amplitude of complex number 0 is not defined.

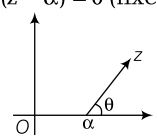
(ii) If $z_1 = z_2 \Leftrightarrow |z_1| = |z_2|$ and amp (z_1) = amp (z_2)

Important Facts Based on Arguments

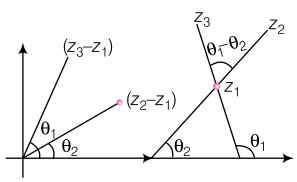
(i) Locus of z, if $\arg(z) = \theta$ is ray excluding origin.



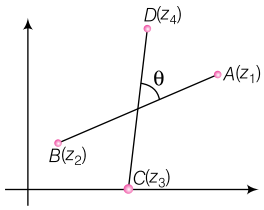
(ii) Locus of z, if $\arg(z - \alpha) = \theta$ (fixed) and $\alpha > 0$ is a ray.



(iii) Angle between two lines $= \theta_1 - \theta_2$



(iv) Angle between line joining z_1, z_2, z_3 and z_4 is $\theta = \arg\left(\frac{z_4 - z_3}{z_2 - z_1}\right)$



IMPORTANT POINTS

(i) $z \cdot \bar{z} = |z|^2$

(ii) $|az_1 + bz_2|^2 + |bz_1 - az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$

(iii) If $z_1, z_2 \neq 0$, then $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \frac{z_1}{z_2}$ is purely imaginary.

(iv) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

(v) $\arg(z_1\bar{z}_2) = \arg(z_1) - \arg(z_2)$;

$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

SPEED BOOSTER

(i) $|z_1 + z_2| = |z_1 - z_2| \Leftrightarrow \arg(z_1) - \arg(z_2) = \frac{\pi}{2}$

(ii) $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow \arg(z_1) = \arg(z_2) = 0$

Triangle Inequalities

It is an important concept to find maximum and minimum values. If z_1 and z_2 are two complex numbers, then

(i) $|z_1 + z_2| \le |z_1| + |z_2|$ i.e. sum of two sides is always greater than third side.

(ii) $|z_1 + z_2| \ge |z_1| - |z_2|$ i.e. difference of two sides is always smaller than third side.

(iii) $||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|$

Polar Form

If $z = x + iy$, $\arg(z) = \theta$, $|z| = r$, then $z = r(\cos\theta + i\sin\theta)$ is known as polar form.

Note Euler's form of complex number z is $re^{i\theta}$, where $|z| = r$ and θ is an argument. θ is positive when it is measured clockwise and is negative when it measured anti-clockwise.

Square Root of Complex Number

Let $z = a + ib$, then the square root of z is

$$\sqrt{z} = \pm\left(\sqrt{\frac{1}{2}\sqrt{a^2 + b^2} + a} + i\frac{1}{2}\sqrt{\frac{1}{2}\sqrt{a^2 + b^2} - a}\right), b > 0$$

$$= \pm\left(\sqrt{\frac{1}{2}\sqrt{a^2 + b^2} + a} - i\frac{1}{2}\sqrt{\frac{1}{2}\sqrt{a^2 + b^2} - a}\right), b < 0$$

Note $\because \quad z = re^{i\theta}$

$\Rightarrow \quad \log_e z = \log_e(r \cdot e^{i\theta})$

$\Rightarrow \quad \log_e z = \log_e r + i\theta\log_e e$

$\Rightarrow \quad \log_e z = \log_e r + i\theta$

Cube Root of Unity

Let $x = (1)^{1/3}$, Then, $x^3 - 1 = 0$

$\Rightarrow \quad (x-1)(x^2 + x + 1) = 0$

$\Rightarrow \quad x = 1,\ x = \dfrac{-1 - i\sqrt{3}}{2} = \omega,\ x = \dfrac{-1 + i\sqrt{3}}{2} = \omega^2$

$\Rightarrow \quad x = 1, \omega, \omega^2$

Properties of Cube Root of Unity

(i) $1 + \omega + \omega^2 = 0$ (ii) $\omega = \overline{\omega}^2,\ \overline{\omega} = \omega^2,\ \omega^3 = 1$

(iii) $\dfrac{1}{\omega} = \omega^2,\ \dfrac{1}{\omega^2} = \omega$

Some Important Results on Cube Root of Unity

(i) $x^2 + xy + y^2 = y^2\left(\dfrac{x^2}{y^2} + \dfrac{x}{y} + 1\right) = y^2\left(\dfrac{x}{y} - \omega\right)\left(\dfrac{x}{y} - \omega^2\right)$

$= (x - \omega y)(x - \omega^2 y)$

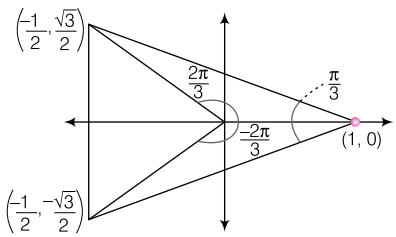
(ii) $x^2 - xy + y^2 = (x + \omega y)(x + \omega^2 y)$

(iii) $\omega^{3n} = 1$

(iv) Sum of pth power of cube root of unity is

$$(1)^p + (\omega)^p + (\omega^2)^p = \begin{cases} \text{if } p \text{ is not multiple of 3} \\ 3, \text{ if } p \text{ is multiple of 3} \end{cases}$$

(v) 3 cube roots of unity when plotted on argand plane constitute the vertices of equilateral triangle.



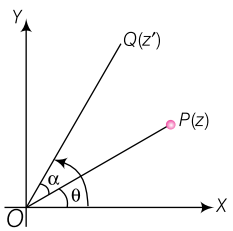
Rotation of Complex Number

Let z be a complex number such that $z = re^{i\theta}$. If z is rotated through an angle α, then

$$z' = re^{i(\theta + \alpha)}$$

$$\Rightarrow \quad z' = re^{i\theta} \cdot e^{i\alpha}$$

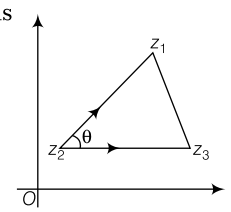
$$\Rightarrow \quad z' = z \cdot e^{i\alpha}$$



Important Applications of Rotation

(i) If $|z| \neq |z'|$. Then, above result is written as $\dfrac{z'}{|z'|} = \dfrac{z}{|z|} e^{i\alpha} z$

(ii) $\dfrac{z_1 - z_2}{|z_1 - z_2|} = \dfrac{z_3 - z_2}{|z_3 - z_2|} e^{i\theta}$



Some Important Geometrical Results

(i) Area of ΔABC with vertices $A(z_1), B(z_2)$ and $C(z_3)$ is given by

$$\Delta = \frac{1}{4}\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix}$$

ΔABC with vertices $A(z_1), B(z_2)$ and $C(z_3)$ is equilateral if and only if

$$\frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} + \frac{1}{z_1 - z_2} = 0$$

$$\Leftrightarrow \quad z_1^2 + z_2^2 + z_3^2 + z_2 z_3 + z_3 z_1 + z_1 z_2 = 0$$

$$\Leftrightarrow \quad \begin{vmatrix} 1 & z_2 & z_3 \\ 1 & z_3 & z_1 \\ 1 & z_1 & z_2 \end{vmatrix} = 0$$

(ii) When a complex number is multiplied by i, it rotates an angle $\dfrac{\pi}{2}$.

(iii) If z_0 is centroid/orthocentre/ circumcentre/incentre and z_1, z_2 and z_3 are vertices of the equilateral triangle, then

$$z_1^2 + z_2^2 + z_3^2 = 3z_0^2$$

Also, if z_0 at origin, then $z_1^2 + z_2^2 + z_3^2 = 0$ i.e. $\Sigma z^2 = 0$

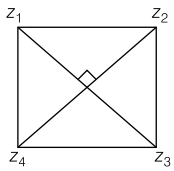
Also, $z_1 z_2 + z_2 z_3 + z_3 z_1 = 0$ i.e. $\Sigma z_1 z_2 = 0$

➦ If z_1, z_2, z_3 and z_4 are vertices of square taken in order, then the following results are true :

(i) $\dfrac{z_1 - z_3}{z_2 - z_4}$ is purely imaginary. (ii) $\dfrac{z_1 - z_2}{z_3 - z_4}$ is purely real.

(iii) $\dfrac{z_4 - z_3}{z_2 - z_3}$ is purely imaginary.

(iv) $\dfrac{z_2 + z_4}{z_1 + z_3} = 1$



De-Moivre's Theorem

It states that $\cos n\theta + i \sin n\theta$ is one of the values of $(\cos\theta + i\sin\theta)^n$, $\forall\ n \in Q$.

If n is an integer, then $(\cos n\theta + i \sin n\theta)$ is the direct value, if n is rational which is not an integer, then it is one of the values of $(\cos\theta + i\sin\theta)^n$.

(i) If n is any rational number, then $(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$.

(ii) If $z = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots(\cos\theta_n + i\sin\theta_n)$, then $z = \cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)$ where, $\theta_1, \theta_2, \ldots, \theta_n \in R$

(iii) If $z = r(\cos\theta + i\sin\theta)$ and n is positive integer, then

$$z^{1/n} = r^{1/n}\left[\cos\left(\frac{2k\pi + \theta}{n}\right) + i\sin\left(\frac{2k\pi + \theta}{n}\right)\right]$$

where, $k = 0, 1, 2, 3, \ldots, (n-1)$

Four Basic Steps to Determine Roots of Complex Number

Let us learn these steps through an example find all four, 4th root of $(1+i)$

***Step* I** Write polar form of $(1+i)$ i.e.

$$1+i=\sqrt{2}\left[\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right]$$

$$(1+i)^{1/4}=2^{1/8}\left[\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right]^{1/4}$$

***Step* II** Add $2n\pi$ to argument

$$=2^{1/8}\left[\cos\left(2n\pi+\frac{\pi}{4}\right)+i\sin\left(2n\pi+\frac{\pi}{4}\right)\right]^{1/4}$$

***Step* III** Apply De-Moivre's theorem

$$(1+i)^{\frac{1}{4}}=2^{\frac{1}{8}}\left[\cos\left(\frac{\frac{\pi}{4}+2n\pi}{4}\right)+i\sin\left(\frac{\frac{\pi}{4}+2n\pi}{4}\right)\right]$$

***Step* IV** Put $n=0, 1, 2, \ldots, (n-1)$

Here, put $n=0, 1, 2, 3$

$$(1+i)^{\frac{1}{4}}=2^{\frac{1}{8}}e^{\frac{i\pi}{16}}, 2^{\frac{1}{8}}e^{\frac{i9\pi}{16}}, 2^{\frac{1}{8}}e^{\frac{i17\pi}{16}}, 2^{\frac{1}{8}}e^{\frac{i25\pi}{16}}$$

Shortcut Method

Observe the question in which product of root is asked.

Que. Find continued product of all 5 values of $(1+i)^{2/5}$.

Sol. Let $(1+i)^{2/5}=z \Rightarrow z^5=(1+i)^2$

$\Rightarrow$ $z^5+0z^4+0z^3+0z^2+0z-(1+i)^2=0$

Hence, product of $z_1z_2z_3z_4z_5=-\left(\frac{-(1+i)^2}{1}\right)$

$=(1+i)^2=1-1+2i=2i$

*n*th Roots of Unity

Let $z^n=1$ be a complex number, then $z=1^{1/n}$, here nth roots of unity are $1, e^{i\frac{2\pi}{n}}, e^{i\frac{4\pi}{n}}, \ldots, e^{\frac{2(n-1)\pi}{n}}$. If $\alpha=e^{\frac{i2\pi}{n}}$, then roots can be written as $1, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^{n-1}$.

Properties of *n*th Roots of Unity

(i) nth roots of unity lie on a circle whose centre is (0, 0) and radius is 1.

(ii) nth roots of unity constitute geometric progression with common ratio $\alpha=e^{i\frac{2\pi}{n}}$ and $\alpha^n=1$.

(iii) Sum of pth power of n, nth roots of unity $1^p+\alpha_1^p+\alpha_2^p+\ldots+\alpha_{n-1}^p$

$$=\begin{cases}0, \text{if } p \text{ is not integral multiple of } n\\ n, \text{if } p \text{ is integral multiple of } n\end{cases}$$

(iv) If $1, \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{n-1}$ are nth roots of unity, then $(1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\ldots(1-\alpha_{n-1})=n$ e.g. $(1-\omega)(1-\omega^2)=3$. Here, $1, \omega, \omega^2$ are cube roots of unity.

(v) $(1+\alpha_1)(1+\alpha_2)(1+\alpha_3)\ldots(1+\alpha_{n-1})=\begin{cases}0, \text{if } n \text{ is even}\\ 1, \text{if } n \text{ is odd}\end{cases}$

e.g. Cube roots of unity are 1, ω, ω^2. Hence, $(1+\omega)(1+\omega^2)=1$ but for fourth root of unity $1, -1, i$ and $-i$, we have $[1+(-1)](1+i)[1+(-i)]=0$

(vi) $1\cdot\alpha_1\cdot\alpha_2\cdot\alpha_3\cdot\ldots\cdot\alpha_{n-1}=\begin{cases}1 & \text{if } n \text{ is odd}\\ -1, & \text{if } n \text{ is even}\end{cases}$

SPEED BOOSTER

If $\cos\alpha+\cos\beta+\cos\gamma=0=\sin\alpha+\sin\beta+\sin\gamma$, then

(i) $\cos3\alpha+\cos3\beta+\cos3\gamma=3\cos(\alpha+\beta+\gamma)$

(ii) $\sin3\alpha+\sin3\beta+\sin3\gamma=3\sin(\alpha+\beta+\gamma)$

(iii) $\cos2\alpha+\cos2\beta+\cos2\gamma=0$

(iv) $\sin2\alpha+\sin2\beta+\sin2\gamma=0$

(v) $\cos(2\alpha-\beta-\gamma)+\cos(2\beta-\gamma-\alpha)+\cos(2\gamma-\alpha-\beta)=0$

(vi) $\sin(2\alpha-\beta-\gamma)+\sin(2\beta-\gamma-\alpha)+\sin(2\gamma-\alpha-\beta)=0$

(vii) $\cos(\alpha-\beta)+\cos(\beta-\gamma)+\cos(\gamma-\alpha)=-\frac{3}{2}$

(viii) All roots of equation $(z+1)^n=z^n$ are collinear in complex plane.

(ix) If z_r, $r=1, 2, 3, \ldots, 2m$, $m\in N$ are roots of equation $1+z+z^2+z^3+\ldots+z^{2m}=0$, then $\sum_{r=1}^{2m}\frac{1}{z_r-1}=-m$.

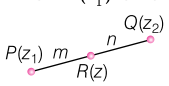
Application of Complex Number in Coordinate Geometry

(i) **Distance between two points** Distance between two points $P(z_1)$ and $Q(z_2)$ is

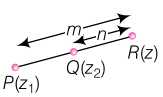
$$PQ=|z_2-z_1|$$

(ii) **Section formulae** Let $R(z)$ divides a join of $P(z_1)$ and $Q(z_2)$ in the ratio $m:n$.

(a) If $R(z)$ divides the line segment PQ internally, then $z=\frac{mz_2+nz_1}{m+n}$.



(b) If $R(z)$ divides the line segment PQ externally, then $z=\frac{mz_2-nz_1}{m-n}$.



IMPORTANT POINTS

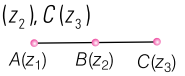
(i) Distance of point $P(z)$ from the origin is $|z|$.

(ii) If $R(z)$ is a mid-point of PQ, then $z=\frac{z_1+z_2}{2}$

(iii) Three points will be collinear, if for $A(z_1), B(z_2), C(z_3)$

$AB+BC=AC$

i.e. $|z_1-z_2|+|z_2-z_3|=|z_1-z_3|$



(iv) The coordinate of centroid of ΔABC whose vertex are $A(z_1), B(z_2)$ and $C(z_3)$, is $G(z)=\frac{z_1+z_2+z_3}{3}$.

Equation of Perpendicular Bisector

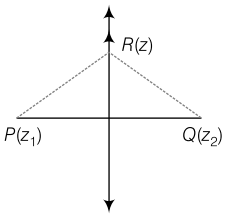
If $P(z_1)$ and $Q(z_2)$ are two fixed points and $R(z)$ is an equidistant point from P and Q, then

$$|z - z_1| = |z - z_2|$$

Also, $|z - z_1|^2 = |z - z_2|^2$

$\Rightarrow (z - z_1)(\bar{z} - \bar{z}_1) = (z - z_2)(\bar{z} - \bar{z}_2)$

$\Rightarrow z(\bar{z}_1 - \bar{z}_2) + \bar{z}(z_1 - z_2) = |z_1|^2 - |z_2|^2$



Equation of Straight Line

Following are equation of straight line in different forms:

(i) **Parametric form** Equation of line joining points $P(z_1)$ and $Q(z_2)$ is $z = t\,z_1 + (1-t)z_2$, where $t \in R$.

(ii) **Non-parametric form** Equation of line joining points $P(z_1)$ and $Q(z_2)$ is $\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$

(iii) **General equation** General equation of straight line is $\bar{a}z + a\bar{z} + b = 0$, where a is a complex number and b is a real number.

IMPORTANT POINTS

(i) Three points z_1, z_2 and z_3 will be collinear, if $\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$

(ii) Slope of line $\bar{a}z + a\bar{z} + b = 0$ is $-\frac{a}{\bar{a}}$.

(iii) If α_1 and α_2 are slopes of two lines in a complex plane, then

(a) lines will be parallel, if $\alpha_1 = \alpha_2$.

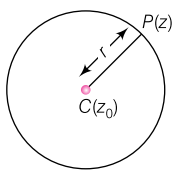
(b) lines will be perpendicular, if $\alpha_1 + \alpha_2 = 0$.

(iv) Slope of line PQ joining two points $P(z_1)$ and $Q(z_2)$ is $\frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$.

(v) Length of perpendicular from a point $P(z_1)$ to the line $\bar{a}z + a\bar{z} + b = 0$ is $\frac{|a\bar{z}_1 + \bar{a}z_1 + b|}{|a| + |\bar{a}|} = \frac{|a\bar{z}_1 + \bar{a}z_1 + b|}{2|a|}$

Equation of a Circle

A circle is the locus of all those points which are equidistant from the given fixed point. This fixed point is called the centre and fixed distance is called radius.



(i) Equation of a circle whose radius is r and centre is $C(z_0)$, is $|z - z_0| = r$. If the centre of the circle lies on the origin, then equation of circle is $|z| = r$.

(ii) The general equation of circle is $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$, where $a \in C$ and $b \in R$. Centre of circle is at $-a$ and radius is $\sqrt{|a|^2 - b}$.

(iii) If $P(z_1)$ and $Q(z_2)$ are the vertices of diameter of circle, then the equation of circle is $(z - z_1)(\bar{z} - \bar{z}_1) + (z - z_2)(\bar{z} - \bar{z}_2) = 0$.

Nature of Circle

The general equation of circle $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$ represents

(i) real circle $|a|^2 > b$ (ii) point circle $|a|^2 = b$

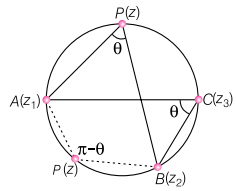
(iii) imaginary circle $|a|^2 < b$

➦ An equation of the circle passing through two points $A(z_1)$ and $B(z_2)$ is

$$(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) + ik \begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0.$$

➦ Equation of circle passing through three non-collinear points is

$$\frac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_1)} = \frac{(\bar{z} - \bar{z}_1)(\bar{z}_3 - \bar{z}_2)}{(\bar{z} - \bar{z}_2)(\bar{z}_3 - \bar{z}_1)}$$



SPEED BOOSTER

(i) If z_1 and z_2 are two fixed points and $k > 0$, $k \neq 1$, is a real number, then $\left|\frac{z - z_1}{z - z_2}\right| = k$ represents circle. If $k = 1$, it represents perpendicular bisector of line segment joining z_1 and z_2.

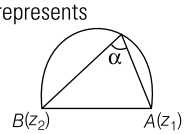
(ii) $|z - z_1| + |z - z_2| = k$, then $k > |z_1 - z_2|$, represents ellipse with foci z_1 and z_2 $k = |z_1 - z_2|$, represents segment joining z_1 and z_2 $k < |z_1 - z_2|$, it does not represent any curve in argand plane.

(iii) $|z - z_1| - |z - z_2| = k$, then $k \neq |z_1 - z_2|$, represents hyperbola $k = |z_1 - z_2|$, represents straight line joining $A(z_1)$ and $B(z_2)$ but excluding the segment AB.

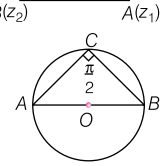
(iv) $|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$, if z_1 and z_2 are two fixed points, then it represents a circle with z_1 and z_2 as extremities of a diameter.

(v) Let z_1 and z_2 be two fixed points and α be a real number such that $0 \le \alpha \le \pi$, then $\arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha$ represents

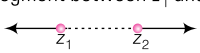
(a) $0 < \alpha < \pi$ represents segment of circle passing through z_1 and z_2.



(b) $\alpha = \frac{\pi}{2}$ represents circle with z_1 and z_2 as diameter as end points.



(c) $\alpha = \pi$ represents straight line joining z_1 and z_2 excluding segment between z_1 and z_2.



(d) $\alpha = 0$ represents the segment joining z_1 and z_2.

Quadratic Equations

An equation of the form $ax^2 + bx + c = 0$, where a, b, c are the set of real numbers and $a \neq 0$, is called quadratic equation. Here, a is called leading coefficient. The quantity $D = b^2 - 4ac$ is known as the discriminant of the equation $ax^2 + bx + c = 0$ and its roots are $x = \frac{-b \pm \sqrt{D}}{2a}$.

SHORT TRICK

In the equation $ax^2 + bx + c = 0$; $a, b, c \in R$

(i) If $a + b + c = 0$, then the roots are 1 and c/a.

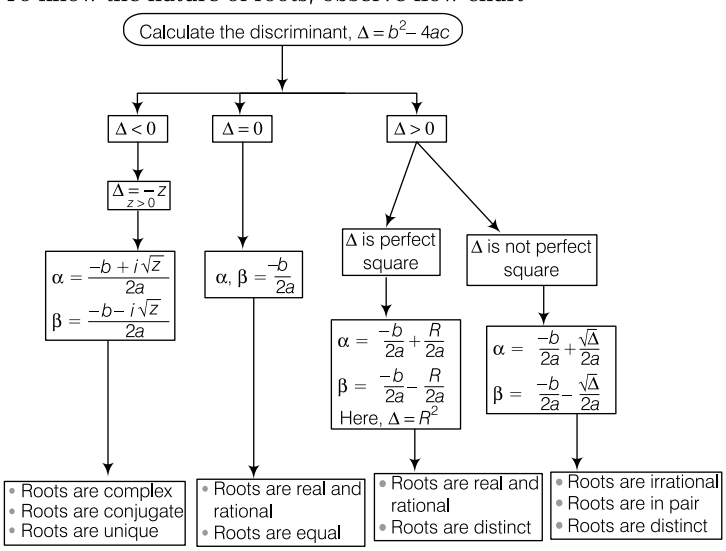
(ii) If $a - b + c = 0$, then the roots are -1 and $\frac{-c}{a}$.

(iii) One root will be reciprocal of the other, if $a = c$.

(iv) Roots are equal in magnitude but opposite in sign, if $b = 0$.

Nature of Roots of $ax^2 + bx + c = 0$

To know the nature of roots, observe flow chart


Calculate the discriminant, Δ = b²– 4ac
Δ < 0
Δ = 0
Δ > 0
Δ = –z, z > 0
α = (–b + i√z)/2a, β = (–b – i√z)/2a
α, β = –b/2a
Δ is perfect square
Δ is not perfect square
α = (–b/2a) + R/2a, β = (–b/2a) – R/2a, Here, Δ = R²
α = (–b/2a) + √Δ/2a, β = (–b/2a) – √Δ/2a
• Roots are complex • Roots are conjugate • Roots are unique
• Roots are real and rational • Roots are equal
• Roots are real and rational • Roots are distinct
• Roots are irrational • Roots are in pair • Roots are distinct

- Irrational and complex roots always occur in pair. If one root is $(a + \sqrt{b})$, then other root will be $(a - \sqrt{b})$. Similarly, if one root is $(a + ib)$, then other root will be $(a - ib)$.
- Both the roots of the equation $(x-a)(x-b) + (x-b)(x-c) + (x-c)(x-a) = 0$ are always real, where $a, b, c \in R$.
- Both the roots of the equation $(x-a)^2 + (x-b)^2 + (x-c)^2 = 0$ are always imaginary, where $a, b, c \in R$.

SPEED BOOSTER

An expression $(a + \sqrt{b})^{f(x)} + (a - \sqrt{b})^{f(x)} = k$, where $k = 2a$ and $(a + \sqrt{b})(a - \sqrt{b}) = 1$, then solve the equation $f(x) = \pm 1$.

e.g. Find the number of real solutions of the equation $(5 + 2\sqrt{6})^{x^2-3} + (5 - 2\sqrt{6})^{x^2-3} = 10$.

By using above, $x^2 - 3 = \pm 1$

$\Rightarrow$ $x^2 - 3 = 1$ and $x^2 - 3 = -1$

$\Rightarrow$ $x^2 = \pm 2 \Rightarrow x = \pm\sqrt{2}$

Hence, the roots are $\pm 2, \pm\sqrt{2}$.

Caution

In case of quadratic equation with real coefficients, imaginary roots always occur in conjugate pairs, however it is not true for quadratic equation with complex coefficients.

Relation between Roots and Coefficients

Let $ax^2 + bx + c = 0$, $a \neq 0$ be a quadratic equation having roots α and β. Then,

$$ax^2 + bx + c \equiv a(x - \alpha)(x - \beta)$$

$$\Rightarrow \quad a\left(x^2 + \frac{bx + c}{a}\right) \equiv a\,[x^2 - (\alpha + \beta)x + \alpha\beta]$$

$$\text{Sum of roots} = \alpha + \beta = \frac{-b}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$$

$$\text{Product of roots} = \alpha\beta = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

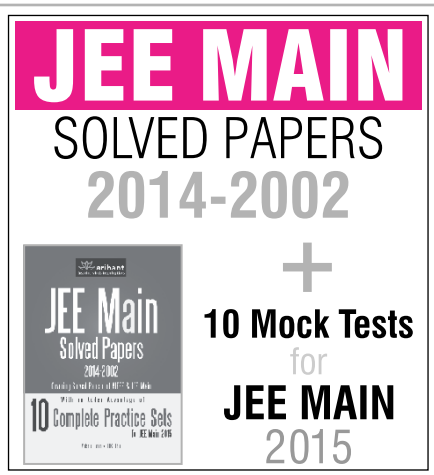
Some Important Shortcuts

(i) If roots of $ax^2 + bx + c = 0$ are in the ratio $m : n$, then $mnb^2 = ac(m + n)^2$.
e.g. Condition for the roots in the ratio $1 : 2$ is $2b^2 = ac(1 + 2)^2 \Rightarrow 2b^2 = 9ac$.

(ii) If one root of $ax^2 + bx + c = 0$ is nth power of the other, then

$$(a^n c)^{\frac{1}{n+1}} + (ac^n)^{\frac{1}{n+1}} + b = 0$$

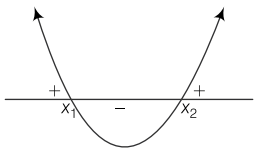
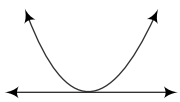
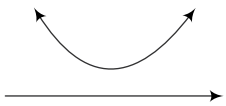
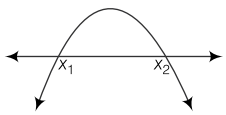
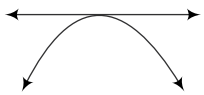
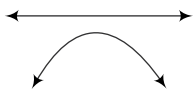
e.g. If one root is square of other root, then $a^2c + ac^2 + b^3 = 3abc$.



Sign of Quadratic Expression and its Graph

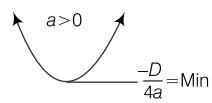
Let $f(x) = ax^2 + bx + c$ or $y = ax^2 + bx + c$

Then, there are following six possible graphs:

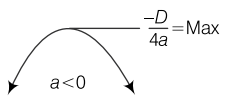
(i) When $a > 0$ and $D > 0$ 	(i) $y > 0$, when $x < x_1$ and $x > x_2$ i.e. graph of the curve is above X-axis. (ii) $y < 0$, when $x \in (x_1, x_2)$ (iii) $y = 0$, when $x = (x_1, x_2)$
(ii) When $a > 0$ and $D = 0$ 	(i) $y \geq 0, \forall\ x \in R$ (ii) $y = 0$ at $x = \frac{-b}{2a}$
(iii) When $a > 0$ and $D < 0$ 	(i) Graph is always above the X-axis i.e. $y > 0, \forall\ x \in R$ (ii) Imaginary roots
(iv) When $a < 0$ and $D > 0$ 	(i) $y > 0$, when $x \in (x_1, x_2)$ (ii) $y = 0$, when $x = (x_1, x_2)$ (iii) $y < 0$, when $x < x_1$ or $x > x_2$
(v) When $a < 0$ and $D = 0$ 	(i) Graph is always below the X-axis i.e. $y \leq 0, \forall\ x \in R$ (ii) $y = 0$ at $x = \frac{-b}{2a}$
(vi) When $a < 0$ and $D < 0$ 	(i) Graph neither touching nor intersecting and lie always below X-axis i.e. $y < 0, \forall\ x \in R$ (ii) Imaginary roots

SPEED BOOSTER

(i) If $a > 0$, then minimum value of $ax^2 + bx + c = 0$ is $\frac{4ac - b^2}{4a}$ or $\frac{-D}{4a}$, where D is discriminant.



(ii) If $a < 0$, then maximum value of $ax^2 + bx + c = 0$ is $\frac{4ac - b^2}{4a}$ or $\frac{-D}{4a}$, where D is discriminant.



(iii) If $x_1, x_2, x_3, \ldots, x_n$ are positive, then least value of $(x_1 + x_2 + x_3 + \ldots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \ldots + \frac{1}{x_n}\right)$ is n^2.

Formation of Quadratic Equation

A quadratic equation whose roots are α and β, is given by

$$(x - \alpha)(x - \beta) = 0$$

$$\Rightarrow \quad x - (\alpha + \beta)x + \alpha\beta = 0$$

$$\Rightarrow \quad x - (\text{sum of roots})\,x + \text{product of roots} = 0 \qquad \ldots(i)$$

PROBLEM SOLVING STRATEGY

TRANSFORMATION OF QUADRATIC EQUATION

***Rule* 1.** To form an equation whose roots are $k(\neq 0)$ times roots of $ax^2 + bx + c = 0$, replace x by $\frac{x}{k}$.

***Rule* 2.** To form an equation whose roots are the negative of the roots in equation, replace x by $-x$.

***Rule* 3.** To form an equation whose roots are reciprocal to given equation, replace x by $\frac{1}{x}$.

***Rule* 4.** To form an equation whose roots are k more than the roots of equation, replace x by $x - k$.

***Rule* 5.** Roots are square roots of given equation, replace x by $\sqrt{x}$.

***Rule* 6.** Roots are cube roots of given equation, replace x by $\sqrt[3]{x}$.

Condition for Common Roots

(i) **One root common** Let α be the common root of quadratic equations $ax^2 + bx + c = 0$ and $a'x^2 + b'x + c' = 0$.

$$\Rightarrow \quad a\alpha^2 + b\alpha + c = 0 \text{ and } a'\alpha^2 + b'\alpha + c' = 0$$

$$\Rightarrow \quad \frac{\alpha^2}{bc' - b'c} = \frac{\alpha}{a'c - c'a} = \frac{1}{ab' - ba'}$$

$$\Rightarrow \quad \alpha = \frac{ca' - c'a}{ab' - ba'} = \frac{bc' - b'c}{a'c - ac'}$$

Hence, condition for common roots is $(ca' - c'a)^2 = (ab' - ba')(bc' - b'c)$

(ii) **Two roots common** If two roots are common, then $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$

Often in JEE and other engineering competitive exams, an unknown constant is asked to find when they have either one common irrational root or an imaginary root.
Since, irrational and imaginary roots always exist in pair, hence both the roots are known, so use the condition $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$.

SPEED BOOSTER

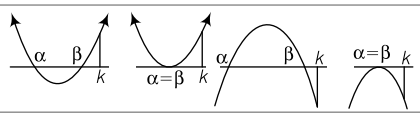
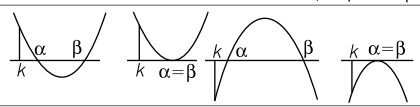
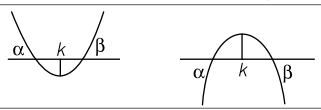
If $a > 0$ and $x = \sqrt{a + \sqrt{a + \sqrt{a + \sqrt{a + \ldots \infty}}}}$, then

$$x = \frac{1 \pm \sqrt{4a + 1}}{2}$$

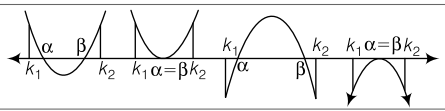
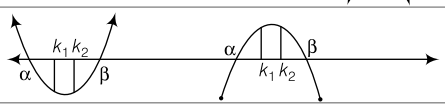
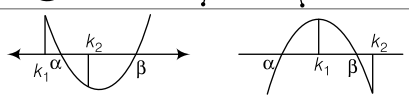
Position of Roots

Let $ax^2 + bx + c = 0$ be α and β as roots, then

Case I Roots with respect to one real number

Situation	Required condition	Graphical situation
(i) $\alpha < \beta < k$, roots are less than k	$D \geq 0$, $af(k) > 0$, $k > \frac{-b}{2a}$	
(ii) $k < \alpha < \beta$, k is less than roots	$D \geq 0$, $af(k) > 0$, $k < \frac{-b}{2a}$	
(iii) $\alpha < k < \beta$, k lies between the roots	$D > 0$, $af(k) < 0$	

Case II Roots with respect to two real numbers k_1 and k_2

Situation	Required condition	Graphical situation
(i) $k_1 < \alpha < \beta < k_2$, roots are between k_1 and k_2	$D \geq 0$, $af(k_1) > 0$, $af(k_2) > 0$, $k_1 < \frac{-b}{2a} < k_2$	
(ii) $\alpha < k_1 < k_2 < \beta$	$D \geq 0$, $af(k_1) < 0$, $af(k_2) < 0$	
(iii) $k_1 < \alpha < k_2 < \beta$	$D \geq 0$, $f(k_1)f(k_2) < 0$	

Higher Degree (*n*th) Polynomials

Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n$

If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the roots of the above equation, then

S_1 = Sum of roots taking one at a time

$$= (-1)^1 \frac{a_1}{a_0} = -\frac{\text{Coefficient of } x^{n-1}}{\text{Coefficient of } x^n}$$

S_2 = Sum of roots taking two at a time

$$= (-1)^2 \frac{a_2}{a_0} = \frac{\text{Coefficient of } x^{n-2}}{\text{Coefficient of } x^n}$$

S_3 = Sum of roots taking three at a time

$$= (-1)^3 \frac{a_3}{a_0} = -\frac{\text{Coefficient of } x^{n-3}}{\text{Coefficient of } x^n}$$

... ...

... ...

Hence, S_r = Sum of roots taking r at a time

$$= (-1)^r \frac{a_r}{a_0} = (-1)^r \frac{\text{Coefficient of } x^{n-r}}{\text{Coefficient of } x^n}$$

SPEED BOOSTER

(i) If $(x - a_1)(x - a_2)(x - a_3)\ldots(x - a_n)$ is the given expression say $p(x)$ and $p(x)$ is a polynomial of degree n, then

Coefficient of x^{n-1}

$= -(a_1 + a_2 + a_3 + \ldots + a_n)$

e.g. Coefficient of x^{99} in

$(x - 1)(x - 2)(x - 3)\ldots(x - 100)$

$= -(1 + 2 + 3 + \ldots + 100)$

$= \frac{-100 \times 101}{2}$

$= -5050$

(ii) Coefficient of x^{n-2}

$= (a_1 + a_2 + a_3 + \ldots + a_n)^2 - \left(\frac{a_1^2 + a_2^2 + \ldots + a_n^2}{2}\right)$

e.g. Coefficient of x^{98} in

$(x - 1)(x - 2)(x - 3)\ldots(x - 100)$

$= (1 + 2 + 3 + \ldots + 100)^2 - \frac{(1^2 + 2^2 + 3^2 + \ldots + 100^2)}{2}$

$= (5050)^2 - 338350$

$= 25164150$

- A polynomial equation of degree n has n roots (real or imaginary).
- If all the coefficients are real, then the imaginary roots occur in conjugate pairs i.e. number of imaginary roots is always even.
- If the degree of polynomial equation is odd, then the number of real roots will also be odd. It follows that atleast one root will be real for odd degree polynomial.
- In equation $f(x) = 0$, where $f(x)$ is a polynomial function and if it has roots $\alpha, \alpha, \beta, \ldots$, where α is a repeated root, then

  $f(x) = (x - \alpha)^2 (x - \beta)\ldots$

  Also, $f'(x) = 2(x - \alpha)[(x - \beta)\ldots] + (x - \alpha)^2[(x - \beta)\ldots]$

  $\Rightarrow x = \alpha$ is also root of $f'(x)$.

  Hence, if $f(x) = 0$ has root α repeated n times, then it will be roots of $f'(x), f''(x), \ldots, f^{(n-1)}(x)$.

Inequality

Basic Facts of Inequalities

The following are very useful point to memorise in inequalities

(i) $x \le y \Rightarrow$ either $x < y$ or $x = y$

(ii) $x < y$ and $y < z \Rightarrow x < z$

(iii) Inequality sign get reversed when both sides are multiplied by negative sign.
i.e. $a < b \Rightarrow -a > -b$

(iv) If $a < b$ and $c < d$, then $a + c < b + d$, also $a - c < b - d$

(v) If $a < b$ and $c > 0$, then $ac > bc$

(vi) If $a < b$ and $c < 0$, then $ac > bc$

(vii) If $0 < a < b$, then $\begin{cases} a^r < b^r, \text{ when } r > 0 \\ a^r > b^r, \text{ when } r < 0 \end{cases}$

(viii) $\begin{cases} a + \frac{1}{a} \ge 2, \forall\ a > 0 \text{ equality exists, if } a = 1 \\ a + \frac{1}{a} \le -2, \forall\ a < 0 \text{ equality exists, if } a = -1 \end{cases}$

Caution

If $a < b$, then $a^2 < b^2$ is not always true. e.g. $2 < 3 \Rightarrow 4 < 9$ but $-4 < 2 \Rightarrow 16 > 9$

Observe the pattern

(i) If $a > 3 \Rightarrow a^2 > 9$ but for $a < 3 \Rightarrow a^2 \ge 0$

(ii) If $2 < a < 4 \Rightarrow 4 < a^2 < 16$
If $-2 < a < 4 \Rightarrow 0 < a^2 < 16$
Similarly, if $-5 \le a \le 3 \Rightarrow 0 \le a^2 \le 25$

Now, we can conclude that $a < b \Rightarrow a^2 < b^2$ follow only when absolute value of a is less than absolute value of b.

Wavy Curve

Let $f(x) = (x - a_1)^{m_1}(x - a_2)^{m_2}(x - a_3)^{m_3} \ldots (x - a_{n-1})^{m_{n-1}}(x - a_n)^{m_n}$

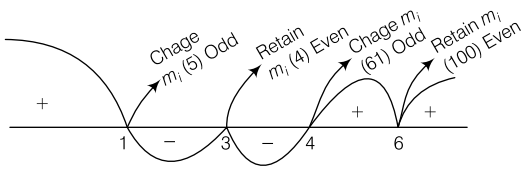
where, $m_1, m_2, m_3, \ldots, m_n \in Z$ and $a_1 < a_2 < a_3 < a_4 < \ldots < a_n$
Then, to find $f(x) > 0$ or $f(x) < 0$, we follow the following algorithm:

***Step* I** Mark the numbers (a_i) in real line in ascending order and assign extreme right as positive sign (i.e. right of a_n).

***Step* II** Sign of left interval will remain same, if m_n is even, otherwise you will alter the sign. This rule is true for other interval as well i.e. you will only alter the sign (positive to negative or negative to positive) when m_i is odd. It will retain, if m_i is even for next interval in left.

***Step* III** If $f(x) > 0$, then consider those interval which are positive and if $f(x) < 0$, then consider those interval in which $f(x)$ is negative.
e.g. $(x-1)^5(x-3)^4(x-4)^{61}(x-6)^{100} > 0$

Find all values of x in the interval for which above inequality is true.



Inequalities based on Absolute Value

(i) $|x| = a \Rightarrow x = \pm a$

(ii) $|x| \le a \Rightarrow -a \le x \le a$

(iii) $|x| \ge a \Rightarrow x \le -a$ or $x \ge a$

(iv) $a \le |x| \le b \Rightarrow a \le x \le b$ or $-b \le x \le -a$

(v) $|f(x) + g(x)| = |f(x)| + |g(x)|$ is equivalent to the system $f(x)g(x) \ge 0$.

Nut Cracker's

(i) If $f(x)$ is a function defined on a closed interval $[a, b]$, such that $f(x)$ is continuous in $[a, b]$, $f(x)$ is derivable on (a, b) and $f(a) = f(b)$, then there exists a point c for which $f'(c) = 0$ i.e. c must be root of $f'(x)$ and lies between a and b.

(ii) If the coefficients of the polynomial equation $f(x) = 0$ have p changes of signs, then the equation $f(x) = 0$ will have atmost p positive roots.

(iii) If the coefficients of polynomial equation $f(-x) = 0$ have q changes of signs, then the equation $f(x) = 0$ will have atmost q negative roots..

(iv) The polynomial equation $f(x) = 0$ will have atmost $(p + q)$ real roots, where p and q are change of signs of coefficients in $f(x)$ and $f(-x)$.

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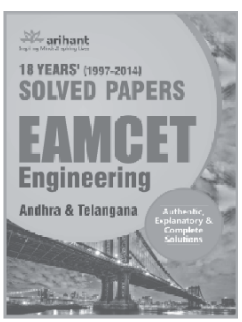

2014-1997
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2015

Sequence and Series

Progression

Sequence A sequence is a function, whose domain is the set of natural numbers. If the range of a sequence is a subset of real numbers (complex numbers), then it is called a real sequence (complex sequence).

If $f: N \to C$ is a sequence we usually denote it as $\{f(1) \cdot f(2) \ldots f(n)\}$

If $t_n = f(n)$, then the sequence is written as $\{t_1, t_2, \ldots, t_n\}$.

Arithmetic Progression

A sequence is said to be arithmetic progression when the difference of two consecutive terms remains constant. This constant is called common difference of an AP and usually denoted by d.

(i) If a is the first term and d is the common difference of an AP, then its nth term t_n is given by $t_n = a + (n-1)d$

(ii) The sum S_n of the first n terms of an AP is given by $S_n = \frac{n}{2}[2a + (n-1)d]$ or $S_n = \frac{n}{2}[a + t_n]$

SHORTCUT

(i) The sum of an AP consisting of odd number of terms = n (middle term), where n is number of terms.

(ii) If a is the first term and d is the common difference of an AP having m terms, then the mth term from the end is $(m-n+1)$th term from the beginning.
Thus, nth term from the end $= a + (m-n)d$

Some Important Facts about an AP

(i) If fixed number is added to (or subtracted from) each term of a given AP, then resulting sequence is also an AP and it has same common difference as that of given AP.

(ii) If each term of an AP is multiplied (or divided) by fixed non-zero constant, then resulting sequence remains in AP with common difference constant times the common difference of original AP.

(iii) If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are two APs, then
(a) $a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots$ are also in AP.
(b) $a_1 - b_1, a_2 - b_2, a_3 - b_3, \ldots$ are also in AP.
(c) $a_1 b_1, a_2 b_2, a_3 b_3, \ldots$ are not in AP.
(d) $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots$ are not in AP.

(iv) If $a_1, a_2, a_3, \ldots, a_n$ are in AP, then
$a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \ldots$
In other words, in an AP, the sum of two terms equidistant from the beginning and end is constant and is equal to sum of first and last terms.

(v) If nth term of a sequence is a linear expression in n, then the sequence is an AP i.e. $T_n = a_n + b$

(vi) If the sum of first n terms of a sequence is a quadratic expression in n, then the sequence is an AP.

(vii) If $a_1, a_2, a_3, \ldots, a_n$ are in AP, then the terms taken at regular intervals from this AP are also in AP.
e.g. $a_2, a_5, a_8, a_{11}, \ldots$

(viii) $a_r = \frac{a_{r-k} + a_{r+k}}{2}$, $k \le n$

PROBLEM SOLVING STRATEGY

(i) Sum of n terms of the series of the form
$\frac{1}{t_1 t_2 \ldots t_k} + \frac{1}{t_2 t_3 \ldots t_{k+1}} + \ldots + \frac{1}{t_n t_{n+1} \ldots t_{n+k-1}}$ is
$S_n = \frac{1}{(k-1)(t_2 - t_1)}\left[\frac{1}{t_1 t_2 \ldots t_{(k-1)}} - \frac{1}{t_{n+1} t_{n+2} \ldots t_{n+k-1}}\right]$

(ii) Sum of n terms of the series of the form
$(t_1 t_2 \ldots t_k) + (t_2 t_3 \ldots t_{k+1}) + \ldots + (t_n t_{n+1} \ldots t_{n+k-1})$ is
$S_n = \frac{1}{(k+1)(t_2 - t_1)}(t_n\ t_{n+1} \ldots t_{n+k} - t_0 t_1 t_2 \ldots t_k)$

Arithmetic Mean

Single Arithmetic Mean

A number A is said to be the single arithmetic mean between two given numbers a and b provided a, A, b are in AP and $A = \frac{a+b}{2}$.

n Arithmetic Mean between Two Terms

Let $A_1, A_2, \ldots, A_n$ be the n arithmetic means between two given numbers a and b. Then, $a, A_1, A_2, A_3, \ldots, A_n, b$ are in AP.

$A_k = a + k, d = a + k\left(\frac{b-a}{n+1}\right)$, where $d = \frac{b-a}{n+1}$, n is number of inserted arithmetic mean and $k = 1, 2, \ldots, n$.

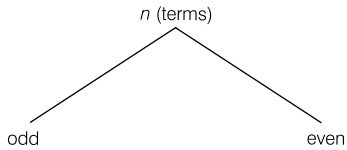
➦ The sum of n arithmetic means between two given numbers is n times the single arithmetic mean between them i.e. if a and b are two numbers and $A_1, A_2, \ldots, A_n$ are n arithmetic mean between them, then

$$A_1 + A_2 + A_3 + \ldots + A_n = n\left(\frac{a+b}{2}\right)$$

or $$\frac{A_1 + A_2 + A_3 + \ldots + A_n}{n} = \frac{a+b}{2}$$

IMPORTANT POINTS

In case if we have to make a choice of few terms which are in AP, then suggested choice is as follows (specially when sum of terms is given):



$a-d, a, a+d$	$a-d, a+d$
$a-2d, a-d, a, a+d,$	$a-3d, a-d, a+d,$
$a+2d$	$a+3d$
$a-3d, a-2d, a-d, a,$	$a-5d, a-3d, a-d,$
$a+d, a+2d, a+3d$	$a+d, a+3d, a+5d$

Arithmetic Mean of *m*th Power

Let $a_1, a_2, \ldots, a_n$ be n positive real numbers (not all equal) and let m be a real number, then

$$\left(\frac{a_1^m + a_2^m + \ldots + a_n^m}{n}\right) > \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^m$$

if $m \in R - [0, 1]$

However, if $m \in (0, 1)$, then

$$\frac{a_1^m + a_2^m + \ldots + a_n^m}{n} < \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^m$$

Geometric Progression

A sequence of numbers (t_n) is said to be in geometric progression (GP), when the first term is non-zero and each successive term is r times the preceding term $\frac{t_n}{t_{n-1}} = r$ is constant for all $n \in N$.

Standard form $a + ar + ar^2 + \ldots + ar^{n-1}$

General term of the series, $T_n = ar^{n-1}$ and sum of all the terms

$$S_n = \begin{cases} a\left(\frac{r^n - 1}{r - 1}\right), \text{ if } r \neq 1 \\ na, \text{ if } r = 1 \end{cases}$$

➦ If the GP is an infinite GP, then

$$S_\infty = \lim_{n \to \infty} a\left(\frac{r^n - 1}{r - 1}\right)$$

$$S_\infty = \begin{cases} \frac{a}{1-r}, \text{ if } & |r| < 1 \\ \text{Does not exist, if } & |r| > 1 \end{cases}$$

Note Infinite GP has definite value only when $|r| < 1$.

Properties of GP

(i) If $a_1, a_2, a_3, \ldots$ are in GP, then

(a) $a_1k, a_2k, a_3k, \ldots$ are also in GP.

(b) $\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k}, \ldots$ are also in GP.

(c) $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots$ are also in GP.

(d) $a_1^k, a_2^k, a_3^k, \ldots$ are also in GP.

(ii) If $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are two GPs, then

(a) $a_1b_1, a_2b_2, a_3b_3, \ldots$ are also in GP.

(b) $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots$ are also in GP.

(c) $a_1 \pm b_1, a_2 \pm b_2, a_3 \pm b_3, \ldots$ are not in GP.

(iii) If $a_1, a_2, a_3, \ldots, a_n$ are in GP, then

(a) $a_1a_n = a_2\ a_{n-1} = a_2\ a_{n-2} = \ldots$

(b) $a_r = \sqrt{a_{r-k} \cdot a_{r+k}}, \quad \leq k \leq n - r$

SHORTCUT

(i) If $a_1, a_2, a_3, \ldots$ are in GP of positive terms, then $\log a_1, \log a_2, \log a_3, \ldots$ are also in AP and *vice-versa*.

(ii) If $a^{n_1}, a^{n_2}, a^{n_3}, \ldots a^{n_m}$ are in GP, then $n_1, n_2, n_3, \ldots, n_m$ will be in AP.

(iii) Product of a GP

***Case* I** If number of terms is odd, then product of terms
$= (\text{middle term})^{\text{Number of terms}}$

***Case* II** If number of terms is even, then product of terms
$= (\text{geometric mean of middle terms})^{\text{Number of terms}}$

Geometric Mean

(i) If a and b are two numbers, then geometric mean of two numbers is $G = \sqrt{ab}$. It is to be noted that the resultant sequence a, G, b is in GP.

(ii) If $a_1, a_2, a_3, \ldots, a_n$ are n numbers, then the geometric mean G of these numbers is given by $G = (a_1a_2a_3 \ldots a_n)^{1/n}$.

(iii) If n numbers $G_1, G_2, G_3, \ldots, G_n$ are said to be geometric means between a and b, if $G_1, G_2, G_3, \ldots, G_n$ are in GP, then

$$G_k = a\left(\frac{b}{a}\right)^{\frac{k}{n+1}}, \text{ where } r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}} \text{ and } k = 0, 1, \ldots, n$$

Key Points

(i) The n numbers $G_1, G_2, G_3, \ldots, G_n$ are n geometric mean between a and b, then $G_1, G_2, G_3, \ldots, G_n = (ab)^{n/2}$

(ii) If A and G are respectively arithmetic and geometric means between two positive numbers a and b, then

(a) $A > G$

(b) the quadratic equation having a, b as roots is $x^2 - 2Ax + G^2 = 0$

(c) two positive numbers are $A \pm \sqrt{A^2 - G^2}$

(d) $\frac{1}{x} + x \geq 2$

Harmonic Progression

The sequence $a_1, a_2, \ldots, a_n, \ldots, (a_i \neq 0)$ is said to be an HP, if the sequence $\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}, \ldots$ is an AP. Then, nth term a_n of an HP is $a_n = \frac{1}{a + (n-1)d}$, where $a = \frac{1}{a_1}$ and $d = \frac{1}{a_2} - \frac{1}{a_1}$

Harmonic Mean

If a and b are two non-zero numbers, then the harmonic mean of a and b is a number H such that the numbers a, H, b are in HP. We have,

$$\frac{1}{H} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right) \Rightarrow H = \frac{2ab}{a+b}$$

Harmonic Mean of n Numbers

If $a_1, a_2, \ldots, a_n$ are n non-zero numbers, then the harmonic mean H of these numbers is given by

$$\frac{1}{H} = \frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_n}\right)$$

Insertion of Harmonic Means between a and b

The n numbers $H_1, H_2, \ldots, H_n$ are said to be n harmonic means between a and b, if $a, H_1, H_2, H_3, \ldots, H_n, b$ are in HP.

i.e. if $\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \ldots, \frac{1}{H_n}, \frac{1}{b}$ are in AP.

$$\Rightarrow \quad \frac{1}{H_n} = \frac{1}{a} + \frac{n(a-b)}{(n+1)ab} = \frac{b+na}{ab(n+1)}$$

Key Points

If A, G and H are respectively AM, GM and HM between a and b both being unequal and positive, then

(i) $G^2 = AH$ i.e. A, G and H are in geometric progression

(ii) $A \geq G \geq H$ equality only hold when all terms are equal.

Method of Difference

Suppose, $a_1, a_2, a_3, \ldots$ is a sequence such that the sequence $a_2 - a_1, a_3 - a_2, \ldots$ is either an AP or GP.

The nth term a_n of this sequence is obtained as follows:

***Step* I** Denote the nth term and sum of the series upto n terms by a_n and S_n. $S_n = a_1 + a_2 + a_3 + \ldots + a_n$

***Step* II** Rewrite the given series with each term shifted by one place to the right

$$S_n = \ldots + a_1 + a_2 + \ldots + a_n$$

***Step* III** Subtract the above two terms of the series, find a_n

$$\Rightarrow 0 = a_1 + (a_2 - a_1) + (a_3 - a_2) + \ldots + (a_n - a_{n-1}) - a_n$$

$$\Rightarrow a_n = a_1 + (a_2 - a_1) + (a_3 - a_2) + \ldots + (a_n - a_{n-1})$$

***Step* IV** Find the sum of the series, $S = \sum_{k=1}^{n} a_k$

Arithmetico-Geometric Progression

If a is the first term, d is the common difference and r is the common ratio, then $a, (a+d)\,r, (a+2d)\,r^2, \ldots$ $a + (n-1)d\,r^{n-1}, \ldots$ is known as AGP.

(i) The nth term of AGP, $T_n = [a + (n-1)\,d]\,r^{n-1}$.

(ii) Sum of first n terms of AGP is

$$S_n = \frac{a}{1-r} + \frac{dr\,(1 - r^{n-1})}{(1-r)^2} - \frac{[a + (n-1)d]r^n}{(1-r)}$$

(iii) Sum of the AGP containing infinite many terms is

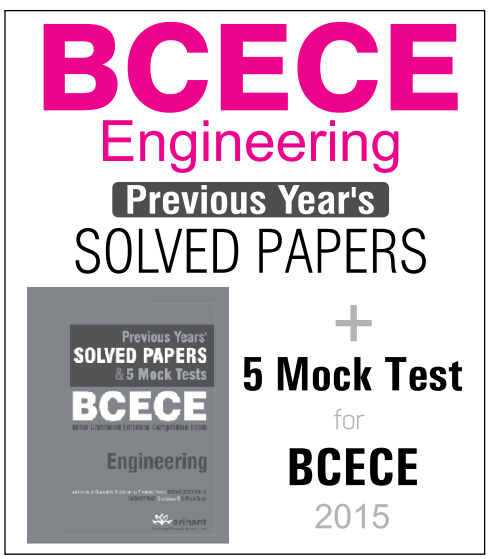
$$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

SPEED BOOSTER

- $1 + 2 + 3 + 4 + \ldots + n = \Sigma n = \frac{n(n+1)}{2}$
- $1^2 + 2^2 + 3^2 + \ldots + n^2 = \Sigma n^2 = \frac{n\,(n+1)\,(2n+1)}{6}$
- $1^3 + 2^3 + 3^3 + \ldots + n^3 = \Sigma n^3 = \left[\frac{n\,(n+1)}{2}\right]^2$
- $1 + 3 + 5 + 7 + \ldots + (2n-1) = (n)^2$
- $2 + 4 + 6 + \ldots + (2n) = n(n+1)$
- $\Sigma k = k \cdot n$
- $1 + x + x^2 + \ldots = (1-x)^{-1}; -1 < x < 1$
- $1 + 2x + 3x^2 + \ldots = (1-x)^{-2}; -1 < x < 1$

To obtain the sum $\sum_{i<j} a_i a_j$, we use the identity

$$2\sum_{i<j} a_i a_j = (a_1 + a_2 + \ldots + a_n)^2 - (a_1^2 + a_2^2 + \ldots + a_n^2)$$



JEE FINAL LAP
QUICK CONCEPT REVIEW
@ CLASS XII SYLLABUS

Let us first revise the concept of set which will help to understand relation as well as probability.

Set

Set is a collection of well-defined objects which are distinct from each other, denoted by the capital letters A, B, C, ...

Methods for Describing a Set

(i) **Listing method** In this method, a set is described by listing element, separated by commas, within braces.

e.g. A = Set of all prime numbers less than 13

$= \{2, 3, 5, 7, 11\}$

(ii) **Set builder method** In this method, instead of listing all elements of a set, we write the property or rule which gives all the elements of the set by that rule

A = Set of all prime numbers less than 13

$= \{x : x \text{ is a prime number}, x < 13\}$

IMPORTANT POINTS

- The order of elements in a set has no importance.
  e.g. {1, 2, 3} and {3, 1,2} are same sets.
- The repetition of elements in a set does not effect the set.
  e.g. {1, 2, 3} and {1, 2, 2, 3} both are the same sets.

Different Types of Sets

(i) **Empty set** A set which has no elements is called an empty set. It is denoted by ϕ or $\{\}$.

e.g. A = Set of all odd numbers divisible by 2

$B = \{x : x \in N \text{ and } 5 < x < 6\}$

➦ If ϕ represents a null set, then ϕ is never written within braces i.e. $\{\phi\}$ is not the null set.

(ii) **Singleton set** A set which has only one element, is called a singleton set.

e.g. $A = \{x : x \in N \text{ and } 7 < x < 9\}$ and $B = \{2\}$

(iii) **Finite and infinite sets** A set in which the process of counting of elements surely comes to an end, is called a finite set. In other words, a set having finite number of elements is called a finite set. Otherwise, it is called infinite set i.e. if the process of counting of elements does not come to an end in a set, then set is called an infinite set.

Note The number of distinct elements contained in a finite set A is called its cardinal number to be denoted by $n(A)$.

(iv) **Equivalent sets** Two finite sets A and B are said to be equivalent, if they have the same number of elements.

(v) **Equal sets** If A and B are two non-empty sets and each element of set A is an element of set B and each element of set B is an element of set A, then sets A and B are called equal sets.

Symbolically, if $x \in A \Rightarrow x \in B$

and $x \in B \Rightarrow x \in A$

➦ Equal sets are equivalent sets while its converse need not be true.

Subset and Superset

Let A and B be two non-empty sets. If each element of set A is an element of set B, then set A is known as subset of set B. If set A is a subset of set B, then set B is called the superset of A. Also, if A is a subset of B, then it is denoted as $A \subseteq B$ and read as A is a subset of B.

Thus, if $x \in A \Rightarrow x \in B$, then $A \subseteq B$

If $x \in A \Rightarrow x \notin B$, then $A \not\subseteq B$

and read as A is not a subset of B.

IMPORTANT POINTS

- Null set is a subset of each set.
- Each set is a subset of itself.
- If A has n elements, then number of subsets of set A is 2^n.

Proper Subset

If each element of A is in set B but set B has atleast one element which is not in A, then set A is known as proper subset of set B.

If A is a proper subset of B, then it is written as '$A \subset B$' and read as 'A is a proper subset of B'.

e.g. $N = \{1, 2, 3, 4, ...\}$

and $I = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$

$\therefore$ $N \subset I$

➦ If A has n elements, then number of proper subsets is $2^n - 1$.

Comparability of Sets

Two sets A and B are said to be comparable, if either $A \subset B$ or $B \subset A$ or $A = B$, otherwise A and B are said to be incomparable.

Universal Set

If there are some sets under consideration, then there happens to be a set which is a superset of each one of the given sets. Such a set is known as the universal set and it is denoted by S or U.

This set can be chosen arbitrarily for any discussion of given sets but after choosing it is fixed.

Power Set

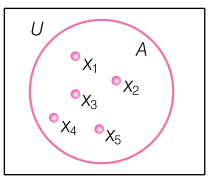
Let A be a non-empty set, then collection of all possible subsets of set A is known as power set. It is denoted by $P(A)$.

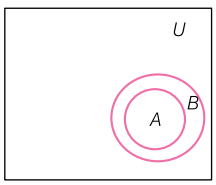
IMPORTANT POINTS

- Each element of a power set is a set.
- Power set of any set is always non-empty set.
- If set A has n elements, then $P(A)$ has 2^n elements.
- If a finite set has n elements. Then, number of subsets having r elements $= {}^nC_r = \dfrac{n!}{r!(n-r)!}$

Venn Diagram

Statements involving sets can be easily understood with pictorial representation of the sets. A set is represented by circle or a closed geometrical figure A, inside the universal set U, which is represented by a rectangular region. Elements of a set A are represented by points within the circle which represents A.





$A \subseteq B$

Operation on Sets

Now, we introduce some operations on sets to construct new sets from the given ones.

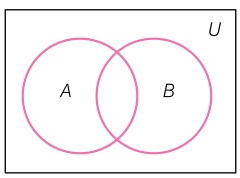
(i) **Union of sets** Let A and B be two sets, then union of A and B is a set of all those elements which are in A or in B or in both A and B. It is denoted by $A \cup B$ and read as A union B.

Symbolically, $A \cup B = \{x : x \in A \text{ or } x \in B\}$

Clearly, $x \in A \cup B \Rightarrow x \in A \text{ or } x \in B$

If $x \notin A \cup B \Rightarrow x \notin A \text{ and } x \notin B$

The Venn diagram of $A \cup B$ is as shown in the figure and the shaded portion represents $A \cup B$.



$A \cup B$

➦ The union of a finite number of sets $A_1, A_2, ..., A_n$ is represented by

$A_1 \cup A_2 \cup A_3 \cup ... \cup A_n$ or $\bigcup_{i=1}^{n} A_i$

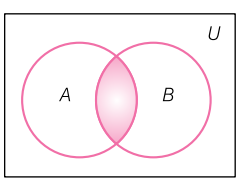
Symbolically, $\bigcup_{i=1}^{n} A_i = \{x : x \in A_i \text{ for atleast one } i\}$

(ii) **Intersection of sets** The intersection of two sets A and B is the set of all the elements which are common in A and B. This set is denoted as $A \cap B$ and read as A intersection B.

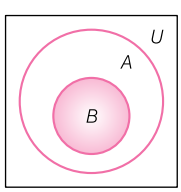
$A \cap B = \{x : x \in A \text{ and } x \in B\} = \{x : x \in A \wedge x \in B\}$

($\wedge$ denotes and)

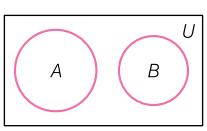
$x \in A \cap B \Leftrightarrow x \in A \text{ and } x \in B$



$A \cap B$ when neither $A \subseteq B$ nor $B \subseteq A$



$A \cap B$ when $B \subseteq A$, $A \cap B = B$



$A \cap B = \phi$ no shaded region when A and B are disjoint sets

➦ The intersection of n sets $A_1, A_2, \ldots, A_n$ is written as

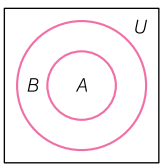
$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$
$$= \{x : x \in A_i, \forall i, 1 \le i \le n\}$$

(iii) **Difference of two sets** The difference of two sets A and B is the set of all those elements of A which are not elements of B. It is denoted by $A - B$.

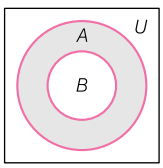
$A - B = \{x : x \in A \text{ and } x \notin B\}$

Thus, $x \in A - B \Rightarrow x \in A$ and $x \notin B$

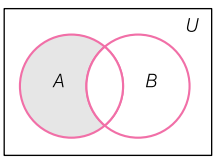
$A - B$ can be represented by Venn diagram (shaded region) as below :



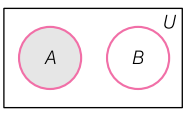
$A - B = \phi$ when $A \subseteq B$



$A - B$ when $B \subseteq A$



$A - B$ when neither $A \subseteq B$ nor $B \subseteq A$

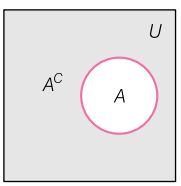


$A - B = A$ when A and B are disjoint sets

(iv) **Complement of a set** Complement of a set A is defined as $U - A$, when U is the universal set and is denoted by

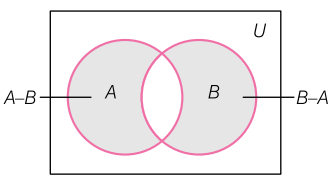
A^C or A' i.e. $A^C = U - A$

or $A^C = \{x : x \in U, x \notin A\}$



➦ $(A^C)^C = A, U^C = \phi, A \cap A^C = \phi, A \cup A^C = U.$

(v) **Symmetric difference** For two sets A and B, symmetric difference is the set $(A - B) \cup (B - A)$ denoted by $A \Delta B$.



Laws of Algebra of Sets

For three sets A, B and C

(i) **Idempotent law**
(a) $A \cup A = A$ (b) $A \cap A = A$

(ii) **Identity law**
(a) $A \cup \phi = A$ (b) $A \cap U = A$

(iii) **Commutative law**
(a) $A \cup B = B \cup A$ (b) $A \cap B = B \cap A$

(iv) **Associative law**
(a) $(A \cup B) \cup C = A \cup (B \cup C)$ (b) $A \cap (B \cap C) = (A \cap B) \cap C$

(v) **Distributive law**
(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(vi) **De-Morgan's law**
(a) $(A \cup B)' = A' \cap B'$ (b) $(A \cap B)' = A' \cup B'$
(c) $A - (B \cap C) = (A - B) \cup (A - C)$
(d) $A - (B \cup C) = (A - B) \cap (A - C)$

(vii) (a) $A - B = A \cap B'$ (b) $B - A = B \cap A'$
(c) $A - B = A \Leftrightarrow A \cap B = \phi$ (d) $(A - B) \cup B = A \cup B$
(e) $(A - B) \cap B = \phi$ (f) $A \cap B \subseteq A$ and $A \cap B \subseteq B$
(g) $A \cup (A \cap B) = A$ (h) $A \cap (A \cup B) = A$

(viii) (a) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$
(b) $A \cap (B - C) = (A \cap B) - (A \cap C)$
(c) $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$
(d) $(A \cap B) \cup (A - B) = A$
(e) $A \cup (B - A) = (A \cup B)$

(ix) (a) $U' = \phi$ (b) $\phi' = U$
(c) $(A')' = A$ (d) $A \cap A' = \phi$
(e) $A \cup A' = U$ (f) $A \subseteq B \Leftrightarrow B' \subseteq A'$

Some Important Results on Order of a Finite Set

If A, B and C are finite sets and U is the finite universal set, then

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$, if $A \cap B \neq \phi$
$= n(A - B) + n(B - A) + n(A \cap B)$

(ii) $n(A \cup B) = n(A) + n(B)$, if $A \cap B = \phi$

(iii) $n(A - B) = n(A) - n(A \cap B) = n(A \cap B') =$ only A

(iv) $n(B - A) = n(B) - n(A \cap B) = n(B \cap A') =$ only B

(v) $n(A - B) + n(B - A) = n(A) + n(B) - 2n(A \cap B)$
$= n(A \cap B') + n(B \cap A')$ or number of elements which belongs to exactly one of A or B.

(vi) Atleast one of the three $= n(A \cup B \cup C) = n(A) + n(B) + n(C)$
$- [n(A \cap B) + n(B \cap C) + n(C \cap A) - n(A \cap B \cap C)]$

(vii) $n(\text{only } A) = n(A \cap B' \cap C') = n[A \cap (B \cup C)']$
$= n(A) - n(A \cap B) - n(A \cap C) + n(A \cap B \cap C)$
the results $n(\text{only } B)$ and $n(\text{only } C)$ are left for the reader.

(viii) None of the elements $A, B, C = n(A' \cap B' \cap C') = n(A \cup B \cup C)'$
$= n(U) - n(A \cup B \cup C)$

(ix) Number of elements exactly two of the three sets A, B, C

$= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$

$= n(A \cap B \cap C') + n(A \cap C \cap B') + n(B \cap C \cap A')$

(x) Number of elements exactly or only one of the sets A, B, C

$= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(C \cap A) + 3n(A \cap B \cap C)$

(xi) Number of elements of A and B but not C given by

$n(A \cap B \cap C') = n(A \cap B) - n(A \cap B \cap C)$

(xii) Number of elements of B and C but not A is given by

$n(B \cap C \cap A') = n(B \cap C) - n(A \cap B \cap C)$

(xiii) Number of elements of C and A but not B is given by

$n(C \cap A \cap B') = n(C \cap A) - n(A \cap B \cap C)$

Note By adding (xi), (xii), (xiii) we get exactly two of the three sets A, B and C which is given in result (ix).

(xiv) $n[(A \cup B)'] = n(A' \cap B') = n(U) - n(A \cup B)$

(xv) $n[(\overline{A \cap B})] = n(\overline{A} \cup \overline{B}) = n(U) - n(A \cap B)$

IMPORTANT POINTS

For any two sets maximum and minimum number of elements in $A \cup B$

By fact $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

- $n(A \cup B)$ will be maximum, if $n(A \cap B)$ is minimum.
- $n(A \cap B)$ will be minimum, if $(A \cap B) = \phi$ as $n(A \cap B) = 0$
- $n(A \cap B)$ will be maximum, if $A \cup B \neq \phi$

$\therefore$ Maximum of $n(A \cup B) = n(A) + n(B)$

Minimum of $n(A \cup B)$ can be determined with the given values of A and B.

Ordered Pair

A pair of objects (elements) taken (listed) in a definite (specific) order, is known as an ordered pair.

In the ordered pair (x_i, y_i), x_i is known as first element and y_i is called second element. Two ordered pair (x, y) and (p, q) are said to be equal $\Rightarrow x = p, y = q$.

➦ Two ordered pairs are equal if and only if their corresponding first elements are equal and second elements are also equal.

Cartesian Product of Sets

For two sets A and B (non-empty sets), the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$ is called cartesian product of the sets A and B, denoted by $A \times B$.

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

If there are three sets A, B, C and $a \in A$, $b \in B$ and $c \in C$, then we form an ordered triplet (a, b, c). The set of all ordered triplets (a, b, c) is called the cartesian product of these sets A, B and C.

i.e. $A \times B \times C = \{(a, b, c) : a \in A, b \in B, c \in C\}$

SPEED BOOSTER

- If A and B contain m and n elements respectively, then there will be $m \times n$ or (mn) elements in $A \times B$ i.e. if $n(A) = m$ and $n(B) = n$, then $n(A \times B) = mn$.
- The cartesian product of A with itself is denoted A^2. So, the number of elements in $A^2 = (m)^2$, when $n(A) = m$. Similarly, the number of elements in $A \times A \times A$ i.e. $n(A^3) = m \times m \times m = m^3$.

Properties of Cartesian Product

(i) $A \times B = \phi$, if either A or B is an empty set.

(ii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(iii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(iv) $A \times (B - C) = (A \times B) - (A \times C)$

(v) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

(vi) If $A \subseteq B$ and $C \subseteq D$, then $(A \times C) \subset (B \times D)$

(vii) If $A \subseteq B$, then $A \times A \subseteq (A \times B) \cap (B \times A)$

(viii) $A \times B = B \times A \Leftrightarrow A = B$

(ix) If either A or B is an infinite set, then $A \times B$ is an infinite set.

(x) $A \times (B' \cup C')' = (A \times B) \cap (A \times C)$

(xi) $A \times (B' \cap C')' = (A \times B) \cup (A \times C)$

(xii) If A and B are any two non-empty sets having n elements in common, then $A \times B$ and $B \times A$ have n^2 elements in common.

(xiii) If $A \neq B$, then $A \times B \neq B \times A$

(xiv) If $A = B$, then $A \times B = B \times A$

(xv) If $A \subseteq B$, then $A \times C = B \times C$ for any set C.

➦ If n elements are common to A and B. Then, number of common elements in $A \times B$ and $B \times A$ are n^2.

Relation (R)

Let X and Y be two non-empty sets. A relation from the set X to set Y is subset of $X \times Y$. If R is a relation from X to Y, then we write $R \subseteq X \times Y$.

If an ordered pair $(x, y) \in R$, then we write xRy.

Domain and Range of Relation

If R is a relation from X to Y, then the set of first entries in distinct ordered pairs in R is called domain of R and set of all second entries in R is called range of R. In other words, R is subset of $X \times Y$. Finally, we say domain $\subseteq X$ and range $\subseteq Y$.

Number of Relations

Let X and Y be two non-empty finite sets having m and n elements, respectively then number of ordered pair in $X \times Y$ is $m \times n$, so the number of subsets of $X \times Y = 2^{m \times n}$

i.e. number of relations from X to $Y = 2^{mn}$

e.g. If $X = \{4, 5, 6\}$, $Y = \{7, 8\}$

then number of relations from X to Y

$=$ number of subsets of $X \times Y = 2^6 = 64$

Inverse Relation

The inverse relation R of a relation R is the set obtained by reversing each of the pairs of R and is denoted by R^{-1}. Thus, if R is a relation from a set A to set B and

$$R = \{(x, y) : x \in A, y \in B\}$$

$$R^{-1} = \{(y, x), y \in B, x \in A\}$$

IMPORTANT POINTS

If any ordered pair (x, y) belongs to R, then the ordered pair (y, x) belongs to R^{-1}. Symbolically, $(x, y) \in R \Rightarrow (y, x) \in R^{-1}$

$\therefore$ Domain R^{-1} = Range R

Range R^{-1} = Domain R

The Inverse of an Inverse Relation

Let R be a relation in $A \times B$, then

$$R = \{(x, y) : x \in A, y \in B\}$$

$\therefore$ $R^{-1} = \{(y, x) : y \in B, x \in A\}$

The inverse of R^{-1} is $(R^{-1})^{-1} = \{(x, y) : x \in A, y \in A\} = R$

Note The inverse of an inverse relation is the original relation.

Types of Relations

Identity Relation

A relation R in a set X is called identity relation in X, if $R = \{(x, x) \mid x \in X\}$. *or*

A relation R is said to be identity relation, if in the set of distinct ordered pairs (x, y), $x \in X$, $y \in Y$ and $x = y$, which means in the set of distinct ordered pairs first and second elements are symmetrical (identical/equal).

Universal Relation

A relation R in a set X is called universal relation, if $R = X \times X$.

i.e. if $X = \{1, 2\}$, then universal relation in $X = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$

Void or Null or Empty Relation in a Set

Let X be a non-empty set, then every subset of $X \times X$ is a relation and empty set, ϕ or $\{\}$ is subset of $X \times X$ is a relation in X such relation is called void or null relation.

Special Types of Relation

Reflexive relation A relation R is said to be reflexive relation, if every element of A is related to itself.

Thus, $(a, a) \in R, \forall\ a \in A \Rightarrow R$ is reflexive.

Symmetric relation A relation R is said to be symmetric relation, iff

$$(a, b) \in R \Rightarrow (b, a) \in R, \forall\ a, b \in A$$

i.e. $aRb \Rightarrow bRa, \forall\ a, b \in A \Rightarrow R$ is symmetric.

Anti-symmetric relation A relation R is said to be anti-symmetric relation, iff

$$(a, b) \in R \text{ and } (b, a) \in R \Rightarrow a = b, \forall\ a, b \in A$$

Transitive relation A relation R is said to be transitive relation iff $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R, \forall\ a, b, c \in A$

Equivalence relation A relation R is said to be an equivalence relation, if it is simultaneously reflexive, symmetric and transitive on A.

Partial order relation A relation R is said to be a partial order relation, if it is simultaneously reflexive, symmetric and anti-symmetric on A.

Total order relation A relation R on a set A is said to be a total order relation on A, if R is a partial order relation on A.

Equivalence Class

Let A be a set and R be an equivalence relation on the set A. If $a \in A$, then the set of all those elements, which satisfy xRa, is called the equivalence class of a, denoted by $[a]$.

Symbolically, the equivalence class $[a] = \{x : x \in A, xRa\}$

IMPORTANT POINTS

- Here, may be other equivalence $[b] = [c] = \ldots$
  The set of all such equivalence classes in the quotient set $\{[a], [b], [c], \ldots\}$ and is denoted by A/R which is read as 'A modulo R' or 'A mod R'..
- If R is an equivalence relation defined on the set A and a and b are any two elements of A, then
  (i) $a \in [a]$, i.e. ...$[a]$ is not empty set (ii) If $b \in [a]$, then $[b] = [a]$
  (iii) $[a] = [b] \Rightarrow (a, b) \in R$
  (iv) Either $[a] \cap [b] = \phi$ or $[a] = [b]$

Composition of Relation

Let R and S be two relations from sets A to B and B to C, respectively, then we can define relation SoR from A to C such that $(a, c) \in SoR \Leftrightarrow \exists\ b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. This relation SoR is called the composition of R and S.

(i) $RoS \neq SoR$ (ii) $(SoR)^{-1} = R^{-1}\ oS^{-1}$

known as **reversal rule**.

IMPORTANT POINTS

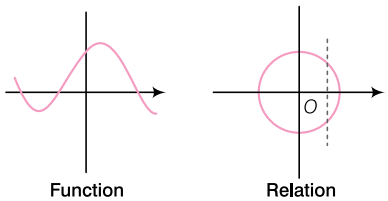
- If R and S are two equivalence relation on a set A, then $R \cap S$ is also an equivalence relation on A.
- The union of two equivalence relations on set AB, then not necessarily an equivalence relation on the set A.
- If R is an equivalence relation on a set A, then R^{-1} is also an equivalence relation on A.
- If a set A has n elements, then number of reflexive relation from A to A is $2^{n^2 - n}$.

Function

A relation R from a set A to a set B is called a function, if

(i) each element of A is associated with some elements of B

(ii) each element of A has unique image in B.



Thus, a function f from a set A to a set B is a subset of $A \times B$ in which each a belonging to A appears in one and only one ordered pair belonging to f. Hence, a function f is a relation from A to B satisfying the following properties. Every function from $A \to B$ satisfies the following conditions

(i) $f \subset A \times B$

(ii) $\forall\ a \in A \Rightarrow (a, f(a)) \in f$ and

(iii) $(a, b) \in f$ and $(a, c) \in f \Rightarrow b = c$

Thus, the ordered pairs of f must satisfy the property that each element of A appears in some ordered pair and no two ordered pairs have same first element.

Note Every function is a relation but every relation is not necessarily function.

Domain, Codomain and Range of a Function

Let $f : A \to B$, then the set A is known as the domain of f and the set B is known as codomain of f. The set of all f images of elements of A is known as the range of f. Thus, Domain of $f = \{a \mid a \in A, (a, f(a)) \in f\}$

Range of $f = \{f(a) \mid a \in A, f(a) \in B\}$.

IMPORTANT POINTS

It should be noted that range is a subset of codomain. If only the rule of function is given, then the domain of the function is the set of those real numbers, where function is defined. For a continuous function, the interval from minimum to maximum value of a function gives the range. (Give domain and range of all common function)

If domain of $f(x)$ is D_1 and domain of $g(x)$ is D_2, then domain of $f(x) + g(x) = D_1 \cap D_2$

Important Types of Functions

Polynomial Function

If a function f is defined by $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}\,x + a_n$, where n is a non-negative integer and $a_0, a_1, a_2, \ldots, a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n.

- A polynomial of degree one with no constant term is called an odd linear function. i.e. $f(x) = ax, a \neq 0$
- There are two polynomial functions, satisfying the relation $f(x) \cdot f(1/x) = f(x) + f(1/x)$. They are (i) $f(x) = x^n + 1$ and (ii) $f(x) = 1 - x^n$, where n is a positive integer.

Algebraic Function

y is an algebraic function of x, if it is a function that satisfies an algebraic equation of the form $P_0(x)\,y^n + P_1(x)\,y^{n-1} + \ldots + P_{n-1}(x)\,y + P_n(x) = 0$, where n is a positive integer and $P_0(x), P_1(x), \ldots$ are polynomials in x. e.g. $x^3 + y^3 - 3xy = 0$ or $y = |x|$ is an algebraic function, since it satisfies the equation $y^2 - x^2 = 0$.

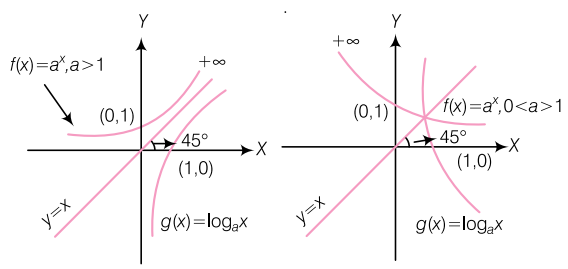
Note All polynomial functions are algebraic but not the converse. A function that is not algebraic is called transcedental function.

Fractional Rational Function

A rational function is a function of the form $y = f(x) = \dfrac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomials and $h(x) \neq 0$

Exponential Function

A function $f(x) = a^x = e^{x \ln a} (a > 0,\ a \neq 1, x \in R)$ is called an exponential function. The inverse of the exponential function is called the logarithmic function.

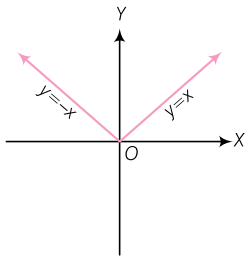


i.e. $g(x) = \log_a x$. For $f(x) = e^x$ domain is R and range is R^+. For $f(x) = e^{\frac{1}{x}}$ domain is $R - \{0\}$ and range is $R^+ - \{1\}$.

Absolute Value Function

A function $y = f(x) = |x|$ is called the absolute value function or modulus function. It is defined as

$$y = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$



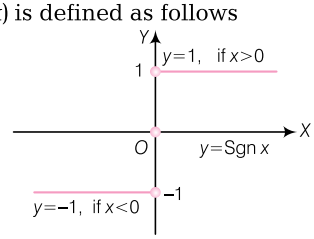
For $f(x) = |x|$, domain is R and range is $R^+ \cup \{0\}$.

For $f(x) = \frac{1}{|x|}$, domain is $R - \{0\}$ and range is R^+.

Signum Function

A function $y = f(x) = \text{Sgn}\,(x)$ is defined as follows

$$y = f(x) = \begin{cases} 1, & \text{for } x > 0 \\ 0, & \text{for } x = 0 \\ -1, & \text{for } x < 0 \end{cases}$$



It is also written as Sgn

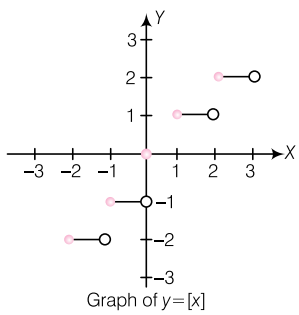
$x = |x|/x;\ x \neq 0;\ f(0) = 0$

Greatest Integer or Step up Function

The function $y = f(x) = [x]$ is called the greatest integer function, where $[x]$ denotes the greatest integer less than or equal to x. Note that for

$-1 \le x < 0; [x] = -1,\ 0 \le x < 1; [x] = 0$

$1 \le x < 2; [x] = 1,\ 2 \le x < 3; [x] = 2$ and so on.



Graph of $y = [x]$

For $f(x) = [x]$, domain is R and range is I.

For $f(x) = \frac{1}{[x]}$, domain is $R - [0, 1)$ and range is $\left\{\frac{1}{n} \middle| n \in I - \{0\}\right\}$.

Properties of Greatest Integer Function

(a) $[x] \le x < [x] + 1$ and $x - 1 < [x] \le x,\ 0 \le x - [x] < 1$

(b) $[x + m] = [x] + m$, if m is an integer.

(c) $[x] + [y] \le [x + y] \le [x] + [y] + 1$

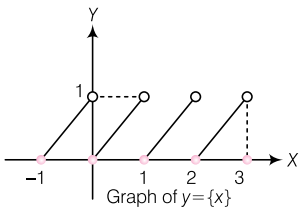
(d) $[x] + [-x] = \begin{cases} 0, \text{if } x \text{ is an integer} \\ -1, \text{otherwise} \end{cases}$

Fractional Part Function

It is defined as $g(x) = \{x\} = x - [x]$

e.g. The fractional part of the number 2.1 is

$2.1 - 2 = 0.1$ and the fractional part of -3.7 is 0.3. The period of this function is 1 and graph of this function is as shown.



Graph of $y = \{x\}$

For $f(x) = \{x\}$, domain is R and range is $[0, 1)$.

For $f(x) = \frac{1}{\{x\}}$, domain is $R - I$ and range is $(1, \infty)$.

Approach to Plot the Graph of Rational Function

(i) Examine whether denominator has a root or not. If no, then graph is continuous and f is non-monotonic.

e.g. $f(x) = \frac{x}{x^2 - 5x + 9}$

If denominator has roots, then $f(x)$ is discontinuous. Such functions can be monotonic/ non-monotonic.

e.g. $f(x) = \frac{x^2 + 2x - 3}{x^2 + 2x - 8} = \frac{(x + 3)(x - 1)}{(x + 4)(x - 2)}$

(ii) If numerator and denominator has a common factor (say $x = a$) it would mean removable discontinuity at $x = a$, e.g. $f(x) = \frac{(x - 2)(x - 1)}{(x + 3)(x - 2)}$. Such a function will always be monotonic i.e. either increasing or decreasing and removable discontinuity at $x = 2$.

(iii) Compute points where the curve crosses the X-axis and also where it cuts the Y-axis by putting $y = 0$ and $x = 0$, respectively and mark points accordingly.

(iv) Compute $\frac{dy}{dx}$ and find the intervals where $f(x)$ is increasing or decreasing and also where it has horizontal tangent.

(v) In regions where curve is monotonic compute y, if $x \to \infty$ or $x \to -\infty$ to find whether y is asymptotic or not.

(vi) If denominator vanishes say at $x = a$ and $(x - a)$ is not a common factor between numerator and denominator, then examine $\lim_{x \to a^-}$ and $\lim_{x \to a^+}$ to find whether f approaches ∞ or $-\infty$. Now, plot the graphs of the following functions.

Equal or Identical Function

Two functions f and g are said to be equal, if

(i) The domain of f = The domain of g

(ii) The range of f = The range of g and

(iii) $f(x) = g(x)$, for every x belonging to their common domain.

e.g. $f(x) = \frac{1}{x}$ and $g(x) = \frac{x}{x^2}$ are identical functions.

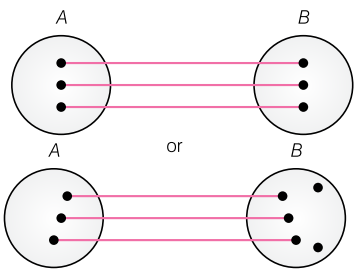
Classification of Functions

One-one Function (injective mapping)

A function $f : A \to B$ is said to be a one-one function or injective mapping, if different elements of A have different f images in B. Thus, for $x_1, x_2 \in A$ and $f(x_1)$, $f(x_2) \in B$, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$.

Examples $R \to R$ $f(x) = x^3 + 1$; $f(x) = e^{-x}$; $f(x) = -\ln x$

Diagrammatically an injective mapping can be shown as



IMPORTANT POINTS

- A continuous function which is always increasing or decreasing in whole domain, then $f(x)$ is one-one.
- If any line parallel to X-axis cuts the graph of the function atmost at one point, then the function is one-one.

Many-one Function (not injective)

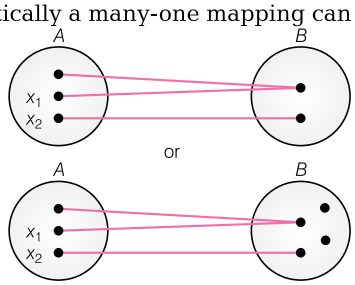
A function $f : A \to B$ is said to be a many-one function, if two or more elements of A have the same f image in B. Thus, $f : A \to B$ is many-one, if for

$$x_1, x_2 \in A, f(x_1) = f(x_2) \text{ but } x_1 \neq x_2$$

Examples $R \to R$ $f(x) = [x]$; $f(x) = |x|$;

$$f(x) = ax^2 + bx + c; f(x) = \sin x$$

Diagrammatically a many-one mapping can be shown as



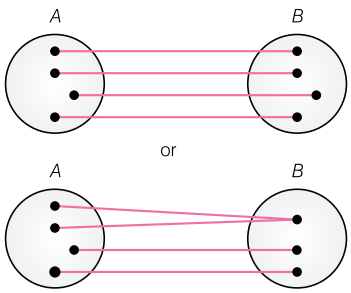
IMPORTANT POINTS

- Any continuous function which has atleast one local maximum or local minimum, then $f(x)$ is many-one. In other words, if a line parallel to X-axis cuts the graph of the function atleast at two points, then f is many-one.
- If a function is one-one, it cannot be many-one and *vice-versa*. One-one + Many-one = Total number of mappings.

Onto Function (surjective mapping)

If the function $f : A \to B$ is such that each element in B (codomain) is the f image of atleast one element in A, then we say that f is a function of A onto B. Thus, $f : A \to B$ is surjective, iff $\forall\ b \in B, \exists$ some $a \in A$ such that $f(a) = b$.

Diagrammatically surjective mapping can be shown as

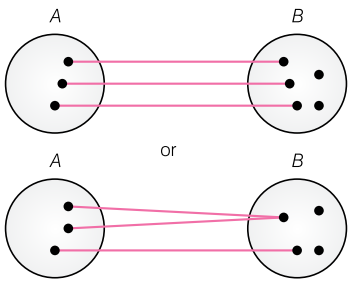


If range = codomain, then $f(x)$ is onto.

Into Function

If $f : A \to B$ is such that there exists atleast one element in codomain which is not the image of any element in domain, then $f(x)$ is into.

Diagrammatically into function can be shown as



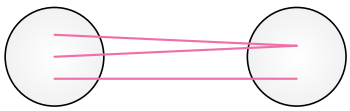
➥ If a function is onto, it cannot be into and *vice-versa*. A polynomial of degree even define from $R \to R$ will always be into and a polynomial of degree odd defined from $R \to R$ will always be onto.

Thus, a function can be one of these four types:

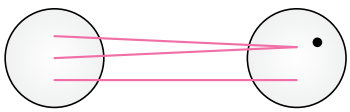
(a) one-one onto (injective and surjective) $(I \cap S)$

(b) one-one into (injective but not surjective) $(I \cap \bar{S})$

(c) many-one onto (surjective but not injective) $(S \cap \bar{I})$



(d) many-one into (neither surjective nor injective) $(\bar{I} \cap \bar{S})$



IMPORTANT POINTS

- If f is both injective and surjective, then it is called a bijective mapping. The bijective functions are also named as invertible, non-singular or biuniform functions.
- If a set A contains n distinct elements, then the number of different functions defined from $A \to A$ is n^n and out of it $n!$ are one-one.

General Definition

Identity Function

The function $f \;\; A \to A$ defined by $f(x) = x, \forall\, x \in A$ is called the identity of A and is denoted by I_A. It is easy to observe that identity function defined on R is a bijection.

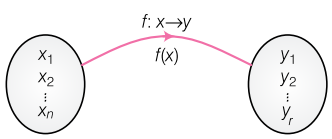
Constant Function

A function $f \;\; A \to B$ is said to be a constant function, if every element of A has the same f image in B. Thus $f \;\; A \to B \; ; \; f\; x \; = c \; , \; \forall\, x \in A, \; c \in B$ is a constant function. Note that the range of a constant function is a singleton and a constant function may be one-one or many-one, onto or into.

Number of Functions

Let $X = \{x_1, x_2, x_3, \ldots, x_n\}$ (i.e. n elements)

and $Y = \{y_1, y_2, y_3, \ldots, y_r\}$ (i.e. r elements)



(i) Total number of functions $= r^n$

$= (\text{Number of elements in codomain})^{\text{Number of elements in domain}}$

(ii) Total number of one to one functions $= \begin{cases} {}^rC_n \cdot n!, & r \geq n \\ 0 \quad , & r < n \end{cases}$

(iii) Total number of many-one functions

$$= \begin{cases} r^n - {}^rC_n \cdot n!, & r \geq n \\ r^n \quad\quad , & r < n \end{cases}$$

(iv) Total number of constant functions $= r$

(v) Total number of onto functions

$$= \begin{cases} r^n - {}^rC_1 (r-1)^n + {}^rC_2 (r-2)^n - {}^rC_3 (r-3)^n + \ldots, & r < n \\ r!, & r = n \\ 0, & r > n \end{cases}$$

(vi) Total number of into functions

$$= \begin{cases} {}^rC_1 (r-1)^n - {}^rC_2 (r-2)^n + {}^rC_3 (r-3)^n - \ldots, & r \leq n \\ r^n, & r > n \end{cases}$$

Composite of Uniformly and Non-uniformly Defined Functions

Let $f : A \to B$ and $g : B \to C$ be two functions. Then, the function $gof : A \to C$ defined by $(gof)(x) = g(f(x)), \forall\, x \in A$ is called the composite of the two functions f and g.

Diagrammatically $\xrightarrow{x}$ $[f]$ $\xrightarrow{f(x)}$ $[g]$ $\longrightarrow g(f(x))$

Thus, the image of every $x \in A$ under the function gof is the g image of the f image of x.

Note gof is defined only, if $\forall\; x \in A$, $f(x)$ is an element of the domain of g, so that we can take its g image. Hence, for the product gof of two functions f and g, the range of f must be a subset of the domain of g. Note that gof in general not equal to fog.

Properties of Composite Functions

- The composite of functions is not commutative i.e. $gof \neq fog$.
- The composite of functions is associative e.g. if f, g, h are three functions such that $fo(goh)$ and $(fog)oh$ are defined, then
  $$fo(goh) = (fog)\, oh$$
  Associativity $f : N \to I_0, f(x) = 2x$
  $$g : \; I_0 \to Q, g(x) = \frac{1}{x}, h : Q \to R, \; h(x) = e^{\frac{1}{x}}$$
  $$(hog)of = ho(gof) = e^{2x}$$
- The composite of two bijections is a bijection i.e. if f and g are two bijections such that gof is defined, then gof is also a bijection.

Homogeneous Functions

A function is said to be homogeneous with respect to any set of variables when each of its terms is of the same degree with respect to those variables.

e.g. $5x^2 + 3y^2 - xy$ is homogeneous in x and y. Symbolically, if $f(tx, ty) = t^n \cdot f(x, y)$, then $f(x, y)$ is homogeneous function of degree n.

Inverse of a Function

Let $f : A \to B$ be a one-one and onto function, then there exists a unique function $g : B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x, \forall\, x \in A$ and $y \in B$. Then, g is said to be inverse of f. Thus, $g = f^{-1} : B \to A = \{(f(x), x) \mid (x, f(x)) \in f\}$.

Properties of Inverse Function

- The inverse of a bijection is unique.
- If $f : A \to B$ is a bijection and $g : B \to A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A and I_B are identity functions on the sets A and B, respectively.

Note The graphs of f and g are the mirror images of each other in the line $y = x$. As shown in the figure given below a point (x', y') corresponding to $y = x^2 (x \geq 0)$ changes to (y', x') corresponding to $y = +\sqrt{x}$, the changed form of $x = \sqrt{y}$.

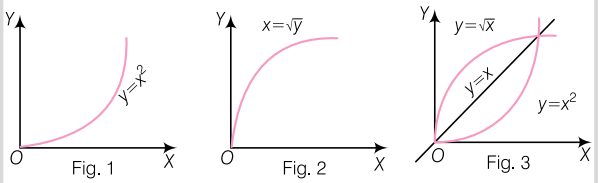


Fig. 1 Fig. 2 Fig. 3

- The inverse of a bijection is also a bijection.
- If f and g are two bijections $f : A \to B, g : B \to C$, then the inverse of gof exists and $(gof)^{-1} = f^{-1}og^{-1}$

Bounded Function

A function is said to be bounded, if $|f(x)| \leq M$, where M is a finite quantity.

Implicit and Explicit Functions

A function defined by an equation not solved for the dependent variable is called an implicit function. e.g. The equation $x^3 + y^3 = 1$ defines y as an implicit function. If y has been expressed in terms of x alone, then it is called an explicit function.

Odd and Even Functions

A function $f(x)$ defined on the symmetric interval $(-a, a)$.

If $f(-x) = f(x)$ for all x in the domain of f, then f is said to be an even function.

e.g. $f(x) = \cos x; g(x) = x^2 + 3$

If $f(-x) = f(y)$ for all x in the domain of f, then f is said to be an odd function.

e.g. $f(x) = \sin x; g(x) = x^2 + x$

Key Feature of Even and Odd Functions

(i) $f(x) - f(-x) = 0 \Rightarrow f(x)$ is even and
$f(x) + f(-x) = 0 \Rightarrow f(x)$ is odd.

(ii) A function may neither be odd nor even.

(iii) Inverse of an even function is not defined and an even function cannot be strictly monotonic.

(iv) Every even function is symmetric about the Y-axis and every odd function is symmetric about the origin.

(v) Every function can be expressed as the sum of an even and an odd function.

e.g. $f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{Even}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{Odd}}$

$$2^x = \frac{2^x + 2^{-x}}{2} + \frac{2^x - 2^{-x}}{2}$$

(vi) The only function which is defined on the entire number line and is even and odd at the same time is $f(x) = 0$.

(vii) If f and g both are even or odd, then the function $f \cdot g$ will be even but if anyone of them is odd, then $f \cdot g$ will be odd.

$f(x)$	$g(x)$	$f(x) + g(x)$	$f(x) - g(x)$
odd	odd	odd	odd
even	even	even	even
odd	even	neither odd nor even	neither odd nor even
even	odd	neither odd nor even	neither odd nor even
$f(x) \cdot g(x)$	$f(x) / g(x)$	$(gof)(x)$	$(fog)(x)$
even	even	odd	odd
even	even	even	even
odd	odd	even	even
odd	odd	even	even

Periodic Function

A function $f(x)$ is called periodic, if there exists a positive number T $(T > 0)$ called the period of the function such that $f(x + T) = f(x)$, for all values of x within the domain of x. e.g. The function $\sin x$ and $\cos x$ both are periodic over 2π and $\tan x$ is periodic over π.

IMPORTANT POINTS

- $f(T) = f(0) = f(-T)$, where T is the period.
- Inverse of a periodic function does not exist.
- Every constant function is always periodic, with no fundamental period.
- If $f(x)$ has a period T and $g(x)$ also has a period T, then it does not mean that $f(x) + g(x)$ must have a period T. e.g. $f(x) = |\sin x| + |\cos x|$.
- If $f(x)$ has a period p, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ also has a period p.
- If $f(x)$ has a period T, then $f(ax + b)$ has a period T / a $(a > 0)$.

General

If x and y are independent variables, then

(i) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k \ln x$ or $f(x) = 0$

(ii) $f(xy) = f(x) \cdot f(y) \Rightarrow f(x) = x^n, n \in R$

(iii) $f(x + y) = f(x) \cdot f(y) \Rightarrow f(x) = a^{kx}$

(iv) $f(x + y) = f(x) + f(y) \Rightarrow f(x) = kx$, where k is a constant.

Transformation of Graphs

(i)	$y = f(x)$ to $y = -f(x)$:	Flip about the X-axis
(ii)	$y = f(x)$ to $y = f(-x)$:	Flip about the Y-axis
(iii)	$y = f(x)$ to $y = f(x) + k$:	Shift the graph $\lvert k \rvert$ units upwards or downwards depending on whether k is positive or negative, respectively.
(iv)	$y = f(x)$ to $y = f(x + k)$:	Advance (shift left) or delay (shift right) the graph by $\lvert k \rvert$ units depending on whether k is positive or negative, respectively.
(v)	$y = f(x)$ to $y = kf(x)$:	Stretch or compress the graph along the Y-axis depending on whether $\lvert k \rvert > 1$ or $\lvert k \rvert < 1$, respectively. Also, flip it about the X-axis, if k is negative (this latter statement follows from part i)
(vi)	$y = f(x)$ to $y = f(kx)$:	Stretch or compress the graph along the X-axis depending on whether $\lvert k \rvert < 1$ or $\lvert k \rvert > 1$, respectively. Also, flip it about the Y-axis, if k is negative (the latter statement follows from part ii)
(vii)	$y = f(x)$ to $y = \lvert f(x) \rvert$:	Reflect the parts of the graph that lie in the lower half (negative parts) into the upper half of the axes.
(viii)	$y = f(x)$ to $y = f(\lvert x \rvert)$:	Discard the left part of the graph (for $x < 0$) and take a reflection of the right part of the graph into the left half of the axes.
(ix)	$y = f(x)$ to $\lvert y \rvert = f(x)$:	Discard the lower part of the graph ($f(x) < 0$) and take a reflection of the upper part of the graph into the lower half of the axes.

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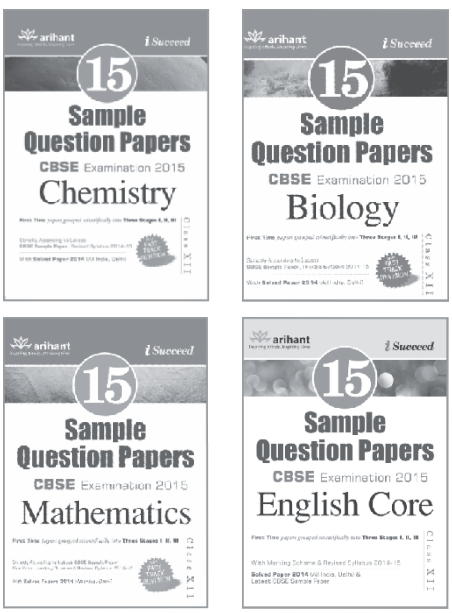

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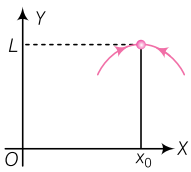
Continuity and Differentiability

Limit of Function

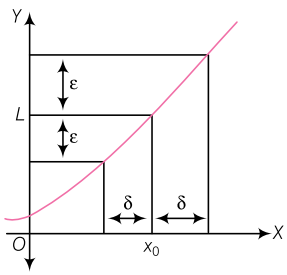
Let $f(x)$ be defined on an open interval about x_0, except possible at x_0 itself. If $f(x)$ gets arbitrarily close to L for all x sufficiently close to x_0, we say that f approaches the limit L as x approaches x_0 and we write $\lim_{x\to x_0} f(x) = L$.



Formal Definition of Limit

Let $f(x)$ be defined on an open interval about x_0, except possible at x_0 itself and let L be a real number.

$\lim_{x\to x_0} f(x) = L$ means that for each real $\varepsilon > 0$ there exists a real $\delta > 0$ such that for all x with $0 < |x - x_0| < \delta$, we have $|f(x) - L| < \varepsilon$ or symbolically, $\forall\ \varepsilon > 0$, $\exists \delta > 0$



$$\forall x\,(0 < |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon)$$

Caution Point

Normally student would have the perception that limit should be a finite number. But it is not really, so it is quite possible that $f(x)$ had infinite limit as $x \to a$. If $\lim_{x\to a} f(x) = \infty$, it would simply mean that functions has tendency to assume to very large positive values in the neighbourhood of $x = a$.

e.g. $\lim_{x\to 0} \frac{1}{|x|} = \infty$.

DO YOU KNOW?

Neighbourhood (*nbd*) of a Point

Let 'x_0' be a real number and let δ be a positive real number. Then, the set of all real numbers lying between $x_0 - \delta$ and $x_0 + \delta$ is called the neighbourhood of x_0 of radius δ and is denoted by $N\delta(x_0)$.

Thus, $N\delta(x_0) = (x_0 - \delta, x_0 + \delta)$
$= \{x \in R \mid x_0 - \delta < x < x_0 + \delta\}$

The set $(x_0 - \delta, x_0)$ is called the left *nbd* of x_0 and the set $(x_0, x_0 + \delta)$ is called the right *nbd* of x_0.

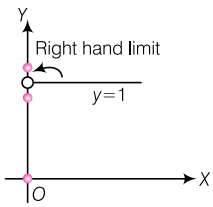
Right Hand Limit

If $\lim_{x\to 0} f(x) = h$ for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x.

$$x_0 < x < x_0 + \delta$$
$$\Rightarrow \quad |f(x) - L| < \varepsilon$$

Then, we called it right hand limit.

e.g. $\lim_{x\to 0^+} \frac{x}{|x|} = +1$



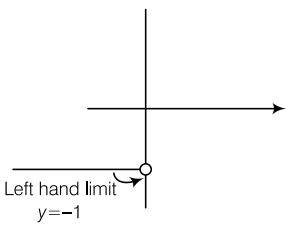
Left Hand Limit

If $\lim_{x\to x_0^-} f(x) = L$ for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x.

$$x_0 - \delta < x < x_0$$
$$\Rightarrow \quad |f(x) - L| < \varepsilon$$

Then, we called it left hand limit.

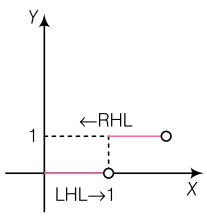
e.g. $\lim_{x\to 0^-} \frac{x}{|x|} = -1$



e.g. $\lim_{x\to 1}[x]$ will not exist as

LHL $= \lim_{x\to 1^-}[x] = 0$,

RHL $= \lim_{x\to 1^+}[x] = 1$



This can be seen graphically as shown in figure.

Existence of Limit

Limit of a function $f(x)$ exists, if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist and both are equal.

Thus, $\lim_{x\to a} f(x)$ exists $\Leftrightarrow \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$

Reasons for non-existence of the limit

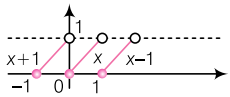
Here, student should aware about reasons for non-existence of limit.

From the discussion, so far it is clear that $\lim_{x\to a} f(x)$ will not exist due to any of these three reasons

(i) $f(x)$ is not defined in the neighbourhood of $x = a$.
e.g. Consider, $\lim_{x\to \pi/2} \sec^{-1}(\sin x)$.

(ii) $f(x)$ does not have a unique tendency.
e.g. Consider, $\lim_{x\to\infty} \cos\left(\frac{1}{x}\right)$.

(iii) Left and right tendencies of $f(x)$ are not same.
e.g. Consider,
$\lim_{x\to 1}\{x\}$, where $\{\}$ denotes fractional part.



➦ For the existence of the limit at $x = a$, $f(x)$ need not be defined at $x = a$. However, if $f(a)$ exists limit need not exist or even, if it exists, then it need not be equal to $f(a)$.

Indeterminancy of Functions

If a unique value cannot be assigned to $f(a)$, then $f(x)$ is said to be indeterminate at $x = a$.

e.g. $f(x) = \dfrac{x^2 - 4}{x - 2}$ is indeterminate at $x = 2$.

$\therefore$ $f(2) = \dfrac{0}{0}$ is indeterminate.

In this section, we list the various type of indeterminancy

Indeterminant Forms

- $\dfrac{\infty_1}{\infty_2} = \dfrac{1/\infty_2}{1/\infty_1} = \dfrac{0}{0}$
- $0 \times \infty = \dfrac{0}{1/\infty} = \dfrac{0}{0}$
- $y = 0^0 \Rightarrow \log y = \log(0^0) \Rightarrow 0 \times \log(0) = 0 \times \infty$
- $y = \infty^0 \Rightarrow \log y = \log(\infty^0) \Rightarrow 0 \times \log(\infty) = 0 \times \infty$
- $y = 1^\infty \Rightarrow \log y = \log(1^\infty) \Rightarrow \infty \times \log(1) = \infty \times 0$
- $\infty_1 - \infty_2$ is also an indeterminate form as the ∞_1 and ∞_2 does not necessarily approach to the same infinity.

Algebra of Limits

Let $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$. If l and m exist, then

(i) $\lim_{x\to a}(f \pm g)(x) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = l \pm m$

(ii) $\lim_{x\to a}(fg)(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x) = lm$

(iii) $\lim_{x\to a}\left(\dfrac{f}{g}\right)(x) = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{l}{m}$, provided $m \neq 0$

(iv) $\lim_{x\to a} k\,f(x) = k \cdot \lim_{x\to a} f(x)$, where k is constant.

(v) $\lim_{x\to a}|f(x)| = \left|\lim_{x\to a} f(x)\right| = |l|$

(vi) $\lim_{x\to a}(f(x))^{g(x)} = \lim_{x\to a} f(x)^{\lim_{x\to a} g(x)} = l^m$

(vii) $\lim_{x\to a} fog(x) = f(\lim_{x\to a} g(x)) = f(m)$, only if f is continuous at $g(x) = m$

In particular,

(a) $\lim_{x\to a} \log f(x) = \log(\lim_{x\to a} f(x)) = \log l$

(b) $\lim_{x\to a} e^{f(x)} = e^{\lim_{x\to a} f(x)} = e^l$

(viii) If $\lim_{x\to a} f(x) = +\infty$ or $-\infty$, then $\lim_{x\to a} \dfrac{1}{f(x)} = 0$.

(ix) If $f(x) \le g(x)$ for every x in the *nbd* of a, then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$.

IMPORTANT POINTS

If $\lim_{x\to c} f(x)g(x)$ exists, then we have following cases:

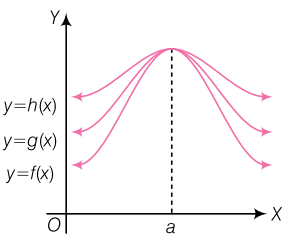
- Both $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist.
- $\lim_{x\to c} f(x)$ exists and $\lim_{x\to c} g(x)$ does not exist.
- Both $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ do not exist.

If $\lim_{x\to c}[f(x) + g(x)]$ exists, then we can have the following cases:

- If $\lim_{x\to c} f(x)$ exists, then $\lim_{x\to c} g(x)$ must exist.
- Both $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ do not exist.

Sandwich Theorem for Evaluating Limits

Let f, g and h be real functions such that $f(x) \le g(x) \le h(x)$ for all x in the common domain of definition. For some real number a, if $\lim_{x\to a} f(x) = l = \lim_{x\to a} h(x)$, then $\lim_{x\to a} g(x) = l$.
This is illustrated in figure.



Some Important Results on Limits

- If $p(x)$ is a polynomial, then $\lim_{x\to a} p(x) = p(a)$.
- $\lim_{x\to 0}(1+x)^{1/x} = e$
- $\lim_{x\to 0} \dfrac{e^x - 1}{x} = 1$
- $\lim_{x\to 0} \dfrac{(1+x)^n - 1}{x} = n$
- $\lim_{x\to 0} \dfrac{\ln(1+x)}{x} = 1$
- $\lim_{x\to 0} \dfrac{\sin^{-1} x}{x} = 1 = \lim_{x\to 0} \dfrac{\tan^{-1} x}{x}$
- $\lim_{x\to 0} \dfrac{\sin x}{x} = \lim_{x\to 0} \dfrac{\tan x}{x} = \lim_{x\to 0} \cos x = 1$ (where, x is in radians)
- $\lim_{x\to\infty}\left(1 + \dfrac{1}{x}\right)^x = e$
- $\lim_{x\to 0} \dfrac{a^x - 1}{x} = \ln(a), a \in R^+$
- $\lim_{x\to a} \dfrac{x^m - a^m}{x^n - a^n} = \dfrac{m}{n} a^{m-n}$
- $\lim_{x\to 0} \dfrac{\log_a(1+x)}{x} = \log_a e, a > 0, \neq 1$

If $\lim_{x\to a} f(x) = 0$, then the following results will be holding true:

- $\lim_{x\to a} \dfrac{\sin f(x)}{f(x)} = \lim_{x\to a} \dfrac{\tan f(x)}{f(x)} = \lim_{x\to a} \cos f(x) = 1$
- $\lim_{x\to a} \dfrac{\sin^{-1} f(x)}{f(x)} = \lim_{x\to a} \dfrac{\tan^{-1} f(x)}{f(x)} = 1$
- $\lim_{x\to a} \dfrac{b^{f(x)} - 1}{f(x)} = \ln b (b > 0)$
- $\lim_{x\to a}(1 + f(x))^{1/f(x)} = e$

Evaluation of Limits

There are many problems which cannot be reduced to any one of the standard limits. In such cases, the following rules may prove useful:

Series Expansion

Following are some of the frequently used series expansions:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$a^x = 1 + (\ln a)\, x + (\ln a)^2 \frac{x^2}{2!} + \ldots, a \in R^+$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots, n \in R. |x| < 1, n \text{ is any real number.}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots, -1 < x \le 1$$

L' HOSPITAL'S RULE

- If $\frac{f(x)}{g(x)}$ has an indeterminancy at $x = 0$ of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$, where f' is called the derivative of f.
- If $\frac{f'(x)}{g'(x)}$ too has an indeterminancy at $x = a$ of the type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then $\lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f''(x)}{g''(x)}$
- This can be done continued till we find a determinate result.

Evaluation of Algebraic Limits

Factorization Method

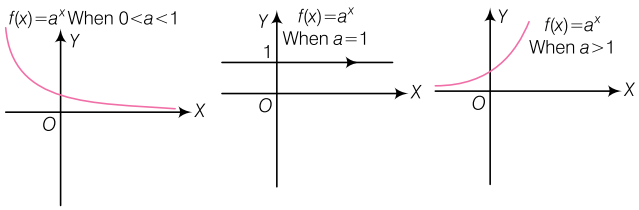
Consider, $\lim_{x\to a}\frac{f(x)}{g(x)}$

If by substituting $x = a, \frac{f(x)}{g(x)}$ reduces to form $\frac{0}{0}$.

Then, we first factorize the $f(x)$ and $g(x)$, then cancel out the common factor to evaluate the limit.

Algebraic Function of ∞ Type

(a) $\frac{\infty}{\infty}$ **form** First we should know the limiting values of $a^x (a > 0)$ as $x \to \infty$. See the graphs of these functions.



Now, see the graph for a^x when $a > 1$. This graph appears to touch X-axis in the negative side of X-axis and thereafter it increases rapidly.

This's why because $\lim_{x\to-\infty} a^x \to 0$, again you will also find the result, $\lim_{x\to\infty} a^x \to \infty$.

Thus, we have $\lim_{x\to\infty} a^x = \begin{cases}\infty, & a > 1\\ 1, & a = 1\\ 0, & \le a < 1\end{cases}$

➦ This type of problem are solved by taking the highest power of the terms tending to infinity as common numerator and denominator. That is after they are cancelled and the rest output is the result or (apply L' Hospital's rule).

(b) $\infty - \infty$ **form** Such problems are simplified (generally rationalised) first, thereafter they generally acquire $\frac{\infty}{\infty}$ form.

Important Points

If m, n are positive integers and a_0, $b_0 \neq 0$ and non-zero real numbers. Then,

$$\lim_{x\to\infty}\frac{a_0x^m + a_1x^{m-1} + \ldots + a_{m-1}x + a_m}{b_0x^n + b_1x^{n-1} + \ldots + b_{n-1}x + b_n} = \begin{cases}0, & m < n\\ \frac{a_0}{b_0}, & m = n\\ \infty, & m > n \text{ when } a_0b_0 > 0\\ -\infty, & m < n \text{ when } a_0b_0 < 0\end{cases}$$

Evaluation of exponential limits of the form 1^∞

To evaluate the exponential form 1^∞, we use the following results:

Result If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, then

$$\lim_{x\to a}\{1 + f(x)\}^{1/g(x)} = e^{\lim_{x\to a}\frac{f(x)}{g(x)}}$$

When $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$

Then, $\lim_{x\to a}\{f(x)\}^{g(x)} = \lim_{x\to a}[1 + f(x) - 1]^{g(x)} = e^{\lim_{x\to a}[f(x)-1]g(x)}$

Remembering Points

(i) $\lim_{x\to 0}(1+x)^{1/x} = e$ (ii) $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e$

(iii) $\lim_{x\to 0}(1+\lambda x)^{1/x} = e^{\lambda}$ (iv) $\lim_{x\to\infty}\left(1+\frac{\lambda}{x}\right)^{1/x} = e^{\lambda}$

Evaluation of limits of form 0^0

When $\lim_{x\to a} f(x) \neq 1$ but $f(x)$ is positive in the neighbourhood of $x = a$.

In this case, we write $\{f(x)\}^{g(x)} = e^{\log_e\{f(x)\}^{g(x)}}$

$\Rightarrow \quad \lim_{x\to a}[f(x)]^{g(x)} = e^{\lim_{x\to a} g(x)\log_e f(x)}$

Summation of Series Using Definite Integral as the Limit

The expression of the form, $\lim_{x\to\infty}\frac{1}{n}\sum_{r=\phi(x)}^{\psi(x)} f\left(\frac{r}{n}\right) = \int_a^b f(x)dx,$

where (i) Σ is replaced by $\int$

(ii) $\frac{r}{n}$ is replaced by x

(iii) $\frac{1}{n}$ is replaced by dx

(iv) To obtain, $a = \lim_{n\to\infty}\frac{\phi(x)}{n}$ and $b = \lim_{n\to\infty}\frac{\psi(x)}{n}$

The value, so obtained is the required sum of the given series e.g.

$\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n}\frac{r}{\sqrt{n^2+r^2}} = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n}\frac{r/n}{\sqrt{1+(r/n)^2}}$

Put $\frac{r}{n} = x, \frac{1}{n} = dx, \sum_{r=1}^{2n} = \int_a^b$

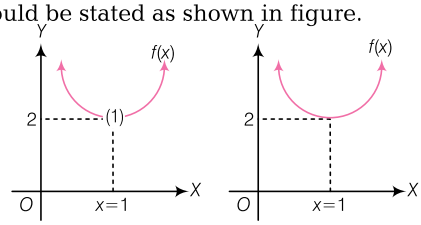
where, $a = \lim_{n\to\infty}\frac{1}{n} = 0$ and $b = \lim_{n\to\infty}\frac{2n}{n} = 2$

$\therefore \quad \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n}\frac{r/n}{\sqrt{1+(r/n)^2}} = \int_0^2\frac{x}{\sqrt{1+x^2}}dx = [\sqrt{1+x^2}]_0^2 = \sqrt{5}-1$

Continuity and Differentiability

A continuous function is a function whose graph can be drawn without lifting the pen from the paper.

As we have discussed on limits, there exist neighbouring points. Graphically, it could be stated as shown in figure.



$\lim_{x\to 1} f(x) = 2$

But when we say that the function $f(x)$ is continuous at a point $x = a$, we mean that at point $(a, f(a))$ the graph of the function has no holes or gaps. That is, its graph is unbroken at a point $(a, f(a))$.

Graphically, it could be stated as shown in figure

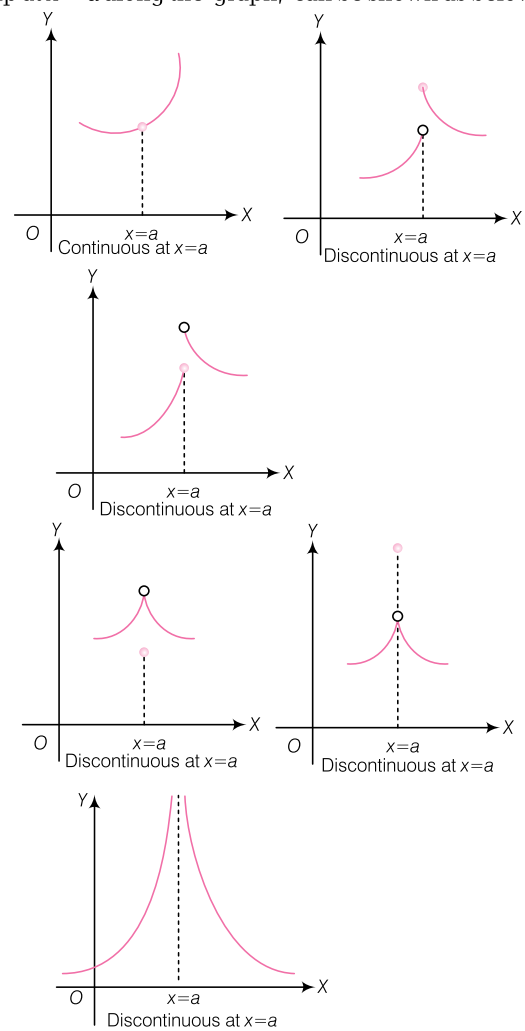
Here, $\lim_{x\to 1} f(x) = 2$ and $f(1) = 2$

Thus, $\lim_{x\to 1} f(x) = f(1)$, hence $f(x)$ is continuous.

➦ The graph of the function is said to be continuous at $x = a$, if while travelling along the graph of the function and in crossing over the point at $x = a$ either from left to right or from right to left one does not have to lift his pen.

Discontinuity

In case one has to lift his pen the graph of the function is said to have a break or discontinuous at $x = a$. Different types of situation, which may come up at $x = a$ along the graph, can be shown as below:



Continuity of a Function

A function $f(x)$ is said to be continuous at $x = a$, if $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$

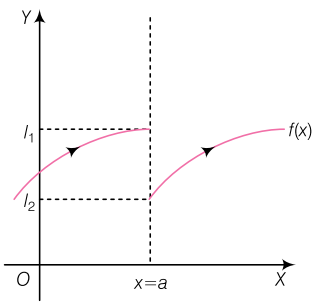
i.e. LHL = RHL = value of the function at

i.e. $\lim_{x\to a} f(x) = f(a)$.

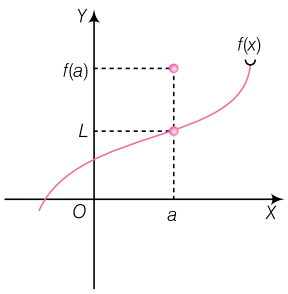
If $f(x)$ is not continuous at $x = a$, we say that $f(x)$ is discontinuous at $x = a$.

$f(x)$ will be discontinuous at $x = a$ in any of the following cases:

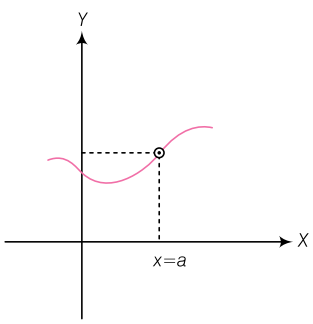
(i) $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist and are not equal.



(ii) $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist and are equal but not equal to $f(a)$.



(iii) $f(a)$ is not defined.



(iv) Atleast one of the limits does not exist.

Caution Point

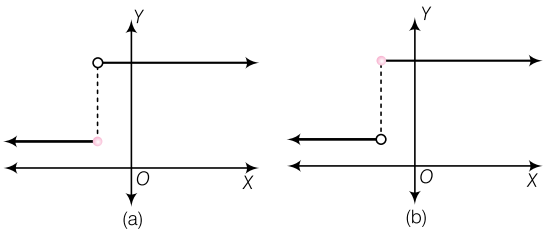
It should be noted that continuity of a function is the property of interval and is meaningful at $x = a$ only if the function has a graph in the immediate neighbourhood of $x = a$, not necessarily at $x = a$. Hence, it should not be mislead that continuity of a function is talked only in its domain.

Directional Continuity

A function may happen to be continuous in only one direction either from the left or from the right.

A right continuous function is a function which is continuous at all points when approached from the right i.e. $c < x < c + \delta$.

Similarly, a left continuous function is a function which is continuous at all points when approached from the left i.e. $c - \delta < x < c$.



Note A function is continuous at $x = a$ if and only if it is both right continuous and left continuous at $x = a$.

Continuity in Interval

Continuity in an Open Interval

A function $f(x)$ is said to be continuous in an open interval (a, b), if it is continuous at each point of (a, b).

Continuity in a Closed Interval

A function $f(x)$ is said to be continuous in a closed interval $[a, b]$, if

(i) $f(x)$ is continuous from right at $x = a$ i.e. $\lim_{x\to a^+} f(x) = f(a)$.

(ii) $f(x)$ is continuous from left at $x = b$ i.e. $\lim_{x\to b^-} f(x) = f(b)$

(iii) $f(x)$ is continuous at each point of the interval (a, b).

Types of Discontinuity

Here, we discuss two types of discontinuity of a function.

Removable Discontinuity

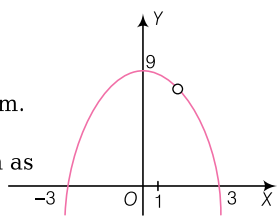
Here, $\lim_{x\to a} f(x)$ necessarily exists, but either is not equal to $f(a)$ or $f(a)$ is not defined. In this case, therefore it is possible to redefine the function in such a manner that $\lim_{x\to a} f(x) = f(a)$ and thus making function continuous.

These discontinuities can be further classified as

(a) **Missing point discontinuity**

e.g. Let $f(x) = \dfrac{(x-1)(9-x^2)}{(x-1)}$, clearly $f(1) \to \dfrac{0}{0}$ form.

$\therefore$ $f(x)$ has missing point discontinuity. Shown as

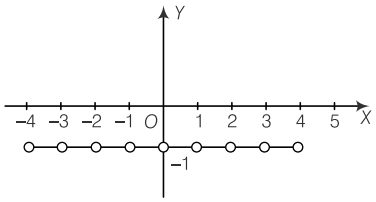


(b) **Isolated point discontinuity**

e.g. Let $f(x) = [x] + [-x]$

$\Rightarrow f(x) = \begin{cases} 0, & x \in I \\ -1, & x \notin I \end{cases}$

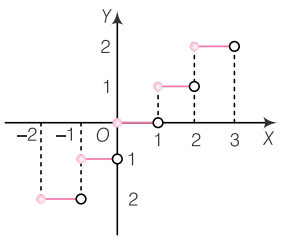
where, x = Integer, has isolated point discontinuity, can be shown as



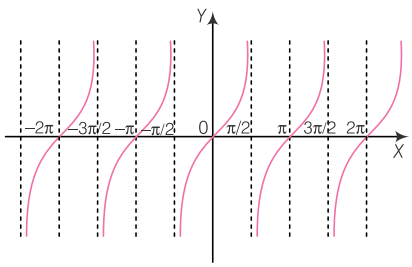
Non-removable Discontinuity

If $\lim_{x \to a} f(x)$ does not exist and therefore it is not possible to redefine the function in any manner and make it continuous. Such discontinuities can further be classified into three types.

(a) **Finite type** (Both limits finite and unequal) Consider, the function $f(x) = [x]$, greatest integer function. As shown in figure the graph has jump of discontinuity at all integral values of x.

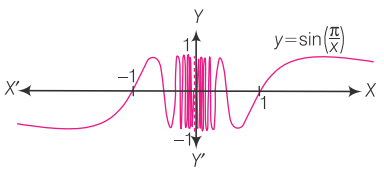


(b) **Infinite type** (Atleast one of the two limits is infinite or vertical asymptote) Consider, the function $f(x) = \tan x$



Here, the function is not defined at points $\pm \frac{\pi}{2}$, $\pm \frac{3\pi}{2}$ and near these points, the function becomes both arbitrarily large and small. Since, the function is not defined at these points, it cannot be continuous.

(c) **Oscillatory** (limit oscillate between two finite quantities) $f(x) = \sin \frac{\pi}{x}$. When $x \to 0, \frac{1}{x} \to \pm\infty$ and $\sin(\to \pm\infty)$ can take any value between -1 to 1 or we can say when $x \to 0$, $f(x)$ oscillates between -1 and 1 as shown in figure.



Theorems on Continuity

Theorem 1 Sum, difference, product and quotient of two continuous functions is always a continuous function. However, $h(x) = \frac{f(x)}{g(x)}$ is continuous at $x = a$ only if $g(a) \neq 0$.

Theorem 2 If $f(x)$ is continuous and $g(x)$ is discontinuous at $x = a$, then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at $x = a$.

Theorem 3 If $f(x)$ and $g(x)$ both are discontinuous at $x = a$, then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at $x = a$.

e.g. $f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$

and $g(x) = \begin{cases} -1, & x \geq 0 \\ 1, & x < 0 \end{cases}$

$\therefore \quad \phi(x) = f(x) \cdot g(x) = -1, \forall\ x \in R$

$\Rightarrow \phi(x)$ is continuous, whereas $f(x)$ and $g(x)$ are discontinuous at $x = 0$.

Continuity of Composite Functions

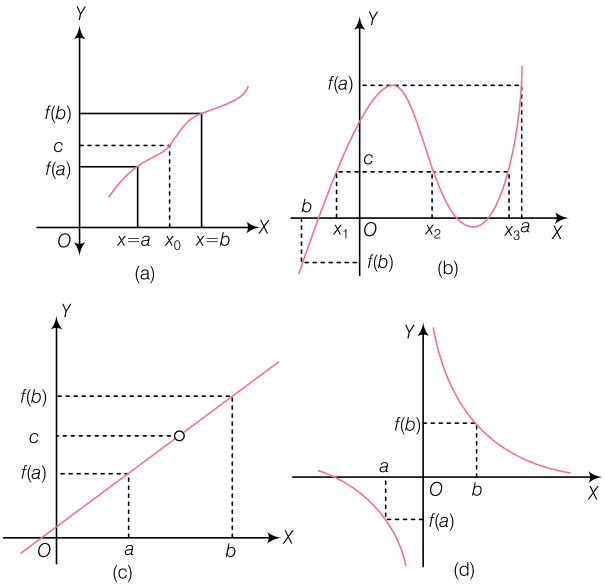
If the function $u = f(x)$ is continuous at the point $x = a$ and the function $y = g(u)$ is continuous at the point $x = f(a)$, then the composite function $y = (gof)(x) = g(f(x))$ is continuous at the point $x = a$.

IMPORTANT POINTS

- If f is continuous on a closed interval $[a, b]$, then it is bounded.
- A continuous function whose domain is some closed interval must have its range also in the closed interval.
- If $f(a)$ and $f(b)$ possess opposite signs, then there exists atleast one solution of the equation $f(x) = 0$ in the open interval (a, b) provided f is continuous in $[a, b]$.
- If f is continuous on $[a, b]$, then f^{-1} is also continuous.

Intermediate Value Theorem

If f is continuous on $[a, b]$ and $f(a) \neq f(b)$, then for any value $c \in (f(a), f(b))$, there is atleast one number x_0 in (a, b) for which $f(x_0) = c$.



From figure (c) and figure (d), it is clear that continuity in interval $[a, b]$ is essential for the validity of this theorem.

Differentiability

Let $f(x)$ be a real-valued function defined on an open interval (a, b), where $c \in (a, b)$.Then, $f(x)$ is said to be differentiable or derivable at $x = c$, if and only if $\lim_{x\to c} \frac{f(x) - f(c)}{(x - c)}$ exists finitely.

This limit is called the derivative or differentiable coefficient of the function $f(x)$ at $x = c$ and is denoted by $f'(c)$ or $Df(c)$ or $\frac{d}{dx}(f(x))_{x=c}$

$$\lim_{x\to c^-} \frac{f(x) - f(c)}{(x - c)} = \lim_{h\to 0} \frac{f(c - h) - f(c)}{-h}$$

is called the left hand derivative of $f(x)$ at $x = c$ and is denoted by $f'(c^-)$ or $Lf'(c)$.

While $\lim_{x\to c^+} \frac{f(x) - f(c)}{(x - c)} = \lim_{h\to 0} \frac{f(c + h) - f(c)}{h}$ is called the right hand derivative of $f(x)$ at $x = c$ and is denoted by $f'(c^+)$ or $Rf'(c)$.

If $Lf'(c) \neq Rf'(c)$, we say that $f(x)$ is not differentiable at $x = c$.

Differentiability in a Set

(i) A function $f(x)$ defined on an open interval (a, b) is said to be differentiable or derivable in open interval (a, b) if it is differentiable at each point of (a, b).

(ii) A function $f(x)$ defined on $[a, b]$ is said to be differentiable or derivable at the end points a and b, if it is differentiable from the right at a and from the left at b. In other words, $\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$ and $\lim_{x\to b^-} \frac{f(x) - f(b)}{x - b}$ both exist.

If f is derivable in the open interval (a, b) and also the end points a and b, then f is said to be derivable in the closed interval $[a, b]$.

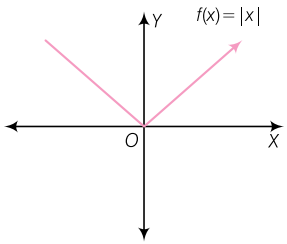
Note A function f is said to be differentiable function, if it is differentiable at every point of its domain.

Relation between Continuity and Differentiability

If a function is differentiable at a point, then it is necessarily continuous at that point. But converse is not necessarily true or $f(x)$ is differentiable at $x = c$.

$\Rightarrow f(x)$ is continuous at $x = c$.

e.g. The function $f(x) = |x|$ is continuous at $x = 0$ but it is not differentiable at $x = 0$, as shown by figure.



Which shows we have sharp edge at $x = 0$. Hence, function is not differentiable but continuous at $x = 0$.

IMPORTANT POINTS

If $f(x)$ is differentiable, then it graph must be smooth i.e. there should be no break or corner. Thus, for a function $f(x)$

- Differentiable $\Rightarrow$ Continuous
- Continuous $\Rightarrow$ May or may not be differentiable
- Not continuous $\Rightarrow$ Not differentiable

How can a function fail to be differentiable

The function $f(x)$ is said to be non-differentiable at $x = a$, if

(i) both $Rf'(a)$ and $Lf'(a)$ exist but are not equal.

(ii) either or both $Rf'(a)$ and $Lf'(a)$ are not finite.

(iii) either or both $Rf'(a)$ and $Lf'(a)$ do not exist.

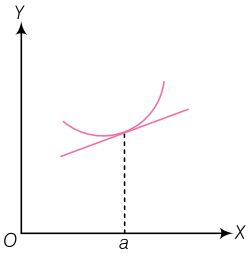
Key Points

If $f(x)$ is a function such that RHD $= f(a^+) = l$
and LHD $= f(a^-) = m$

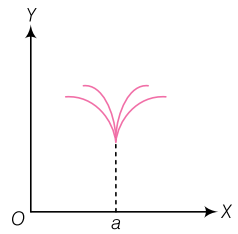
Case I If $l = m =$ some finite value, then the function $f(x)$ is differentiable as well as continuous.

Case II If $l \neq m =$ but have some finite value, then the function $f(x)$ will not be differentiable but it will be continuous.

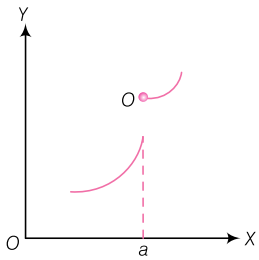
Case III If atleast one of the l or m is infinite, then the function is non-differentiable but we cannot say about continuity of $f(x)$.



Continuous and differentiable



Continuous but not differentiable



Neither continuous nor differentiable

Theorems of Differentiability

Theorem 1 If $f(x)$ and $g(x)$ are both derivable at $x = a$, $f(x) \pm g(x)$, $f(x) \cdot g(x)$ and $\frac{f(x)}{g(x)}$ will also be derivable at $x = a$ $\left\{\text{only if } g(a) \neq 0 \text{ for } \frac{f(x)}{g(x)}\right\}$.

Theorem 2 If $f(x)$ is derivable at $x = a$ and $g(x)$ is not differentiable at $x = a$, then $f(x) \pm g(x)$ will not be derivable at $x = a$.

Theorem 3 If both $f(x)$ and $g(x)$ are non-derivable, then nothing can be said about the sum/difference/product function.

e.g. $f(x) = \sin|x|$, not derivable at $x = 0$

$g(x) = |x|$, not derivable at $x = 0$

Then, the function

$F(x) = \sin|x| + |x|$, not derivable at $x = 0$

$G(x) = \sin|x| - |x|$, derivable at $x = 0$

Theorem 4 If $f(x)$ is derivable at $x = a$ and $f(a) = 0$ and $g(x)$ is continuous at $x = a$.Then, the product function $F(x) = f(x) \cdot g(x)$ will be derivable at $x = a$.

Note If $y = f(x)$ is differentiable at $x = a$, then it is not necessary that the derivative is continuous at $x = a$.

Differentiation

The rate change of a quantity y with respect to another quantity x is called the derivative or differential coefficient of y with respect to x. It is denoted by $\frac{dy}{dx}$.

Some Differentiation Formulae

(i) If $f(x)$ and $g(x)$ are differentiable functions and $\alpha \in R$, then $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$ and $\frac{d}{dx}[\alpha f(x)] = \alpha \frac{d}{dx} f(x)$

(ii) **Product rule** If $f(x)$ and $g(x)$ are differentiable functions, then $\frac{d}{dx}[f(x) \cdot g(x)] = g(x)\frac{d}{dx}[f(x)] + f(x)\frac{d}{dx}[g(x)]$

(iii) **Quotient rule**

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

(iv) **Chain rule** If $y = h(u)$ and $u = f(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

➥ If at all points of a certain interval, $f'(x) = 0$, then the function $f(x)$ preserves a constant value within this interval. This function is said to be constant function.

Differentiation of Some Elementary Functions

(i) The trigonometric functions have the following derivatives

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\cot x) = -\text{cosec}^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\text{cosec}\, x) = -\text{cosec}\, x \cot x$$

(ii) If $f(x)$ is a differentiable function $\frac{d}{dx}(x^n) = nx^{n-1}$ and in general using chain rule $\frac{d}{dx}[f(x)^n] = n[f(x)]^{n-1} f'(x)$.

(iii) If $f(x)$ is a differentiable function $\frac{d}{dx}(\log x) = \frac{1}{x}$ and in general $\frac{d}{dx}[\log f(x)] = \frac{1}{f(x)} f'(x)$

(iv) $\frac{d}{dx}(a^x) = a^x \log a$.

In particular, $\frac{d}{dx}(e^x) = e^x$ and $\frac{d}{dx}(e^{f(x)}) = e^{f(x)} f'(x)$.

(v) $\frac{d}{dx}(\sin^{-1}x) = -\frac{d}{dx}(\cos^{-1}x) = \frac{1}{\sqrt{1-x^2}}$ for $-1 < x < 1$. At the points $x = \pm 1$, $\sin^{-1}x$ and $\cos^{-1}x$ are not differentiable.

(vi) $\frac{d}{dx}(\tan^{-1}x) = -\frac{d}{dx}(\cot^{-1}x) = \frac{1}{1+x^2}$ for all $x \in R$

(vii) $\frac{d}{dx}(\sec^{-1}x) = -\frac{d}{dx}(\text{cosec}^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}}$ for $|x| > 1$.

Differentiation of Composite Functions

If f and g are differentiable functions in their domain, then fog is also differentiable and

$$(fog)'(x) = f'\{g(x)\}\, g'(x).$$

Differentiation Using Substitution

In order to find differential coefficients of complicated expression involving inverse trigonometric functions some substitutions are very helpful, which are listed below.

S.No.	Function	Substitution
1.	$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ or $a\cos\theta$
2.	$\sqrt{a^2 + x^2}$	$x = a\tan\theta$ or $a\cot\theta$
3.	$\sqrt{x^2 - a^2}$	$x = a\sec\theta$ or a cosec θ
4.	$\sqrt{a + x}$ and $\sqrt{a - x}$	$x = a\cos 2\theta$
5.	$a\sin x + b\cos x$	$a = r\cos\alpha,\ b = r\sin\alpha$
6.	$\sqrt{x - \alpha}$ and $\sqrt{\beta - x}$	$x = \alpha\sin^2\theta + \beta\cos^2\theta$
7.	$\sqrt{2ax - x^2}$	$x = a(1 - \cos\theta)$

Differentiation of Implicit Functions

If $f(x, y) = 0$, differentiate with respect to x and collect the terms containing $\frac{dy}{dx}$ at one side and find $\frac{dy}{dx}$.

Differentiation of Parametric Functions

If $x = f(t), y = g(t)$, where t is parameter, then

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} = \frac{\frac{d}{dt}g(t)}{\frac{d}{dt}f(t)} = \frac{g'(t)}{f'(t)}$$

Logarithmic Differentiation Function

The process of taking logarithms of both sides and then differentiating them is called logarithmic differentiation. It is useful in the following cases:

(i) If the given function consists of three or more factors which are function of x.

(ii) If the given function is of the form $[f(x)]^{h(x)}$.

(iii) If the given function is of the form $\frac{h_1(x)h_2(x)}{h_3(x)h_4(x)}$.

Higher Order Differentiation

Let $y = f(x)$ be a differentiable function such that $z = f'(x)$ is also differentiable. Then, the second derivative of $y = f(x)$ is denoted by $y_2(x)$, $f''(x)$ or $\frac{d^2y}{dx^2}$ and is defined by

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

Similarly, we can define

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = f'''(x)$$

and $$f^{(iv)}(x) = \frac{d^4y}{dx^4}$$

or in general,

$$\frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) \text{ for any } n \in N.$$

Note Most of the functions like, polynomial, trigonometric, exponential and logarithmic functions are differentiable at each point of their domain.

Leibnitz Theorem and *n*th Derivatives

Let $u(x)$ and $v(x)$ be functions possessing derivatives upto nth order. Then,

$$(uv)_n = u_n(x)v(x) + {}^nC_1u_{n-1}(x)v_1(x) + \ldots + {}^nC_ku_{n-k}(x)v_k(x) + \ldots + {}^nC_nu(x)v_n(x)$$

where, $u_k(x) = \frac{d^k}{dx^k}u(x)$ [kth derivative of the function $u(x)$], $1 \le k \le n$.

n derivatives of some elementary functions are

$$\frac{d^n}{dx^n}(x^m) = \frac{m!\,x^{m-n}}{(m-n)!}, \text{ if } n \le m$$

and $$\frac{d^n}{dx^n}(x^m) = 0, \text{ if } n > m.$$

$$\frac{d^n}{dx^n}(\sin x) = \sin\left(x + \frac{n\pi}{2}\right),\ \frac{d^n}{dx^n}(\cos x) = \cos\left(x + \frac{n\pi}{2}\right)$$

$$\frac{d^n}{dx^n}(e^{mx}) = m^n\, e^{mx}$$

Speed Booster

(i) $\frac{d^n}{dx^n}[\sin(ax + b)] = a^n \sin\left(\frac{n\pi}{2} + ax + b\right)$

(ii) $\frac{d^n}{dx^n}[\cos(ax + b)] = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$

(iii) $\frac{d^n}{dx^n}(ax + b)^m = \frac{m!}{(m-n)!}a^n(ax + b)^{m-n}$

(iv) $\frac{d^n}{dx^n}[\log(ax+b)] = \frac{(-1)^{n-1}(n-1)a^n}{(ax+b)^n}$

(v) (a) $\frac{d^n}{dx^n}[e^{ax}\sin(bx+c)] = r^n e^{ax}\sin(bx+c+n\phi)$

(b) $\frac{d^n}{dx^n}[e^{ax}\cos(bx+c)] = r^n e^{ax}\cos(bx+c+n\phi)$

where, $r = \sqrt{a^2+b^2}$

and $\phi = \tan^{-1}\left(\frac{b}{a}\right)$

Differentiation of a Determinant

To differentiate a determinant, we differentiate one row (or column) at a time, keeping others unchanged.

e.g. If $\Delta(x) = \begin{vmatrix} f(x) & g(x) \\ u(x) & v(x) \end{vmatrix}$, then

$$\frac{d}{dx}[\Delta(x)] = \begin{vmatrix} f'(x) & g'(x) \\ u(x) & v(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) \\ u'(x) & v'(x) \end{vmatrix}$$

Note Similar results hold for the differentiation of determinants of higher order.

IMPORTANT POINTS

- If $y = f(x)^{\{f(x)\}^{\cdots\infty}}$, then $\frac{dy}{dx} = \frac{y^2 f'(x)}{f(x)\{1-y\log f(x)\}}$
- If $e^{g(y)} - e^{-g(y)} = 2f(x)$, then $\frac{dy}{dx} = \frac{f'(x)}{g'(y)} \cdot \frac{1}{\sqrt{1+\{f(x)\}^2}}$
- If $y = \sqrt{\frac{1+g(x)}{1-g(x)}}$, then $\frac{dy}{dx} = \frac{g'(x)}{[1-g(x)]^2} \cdot \sqrt{\frac{1-g(x)}{1+g(x)}}$
- $y = \sqrt{f(x)+\sqrt{f(x)+\sqrt{f(x)+\ldots\infty}}}$, then $\frac{dy}{dx} = \frac{f'(x)}{2y-1}$
- If $\{f(x)\}^{g(y)} = e^{f(x)-g(y)}$, then $\frac{dy}{dx} = \frac{f'(x)\log(x)}{g'(y)\cdot\{1+\log f(x)\}^2}$
- If $\{f(x)\}^{g(y)} = \{g(y)\}^{f(x)}$, then $\frac{dy}{dx} = \frac{g(y)}{f(x)} \cdot \frac{f'(x)}{g'(y)}\left[\frac{f(x)\log g(y) - g(y)}{g(y)\log f(x) - f(x)}\right]$

Rolle's Theorem

Let f be a real-valued function defined in the closed interval $[a, b]$, such that

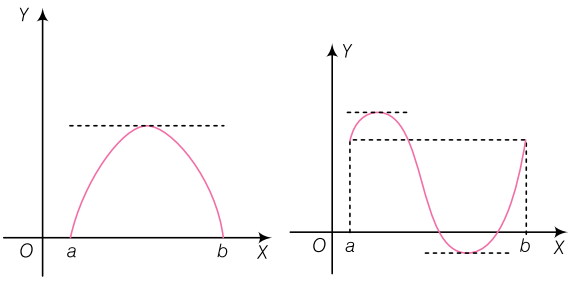
(i) f is continuous in the closed interval $[a, b]$.

(ii) $f(x)$ is differentiable in the open interval (a, b).

(iii) $f(a) = f(b)$

Then, there is some point c in the open interval (a, b), such that $f'(c) = 0$.

Geometrically Under the assumptions of Rolle's theorem, the graph of $f(x)$ starts at point $(a, 0)$ and ends at point $(b, 0)$ as shown in figures.



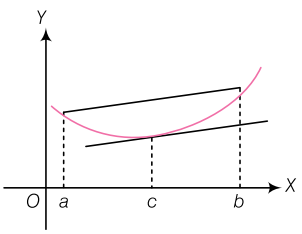
The conclusion is that there is atleast one point c between a and b, such that the tangent to the graph at $(c, f(c))$ is parallel to the X-axis.

➦ Between any two roots of a polynomial $f(x)$, there is always a root of its derivative $f'(x)$.

Lagrange's Mean Value Theorem

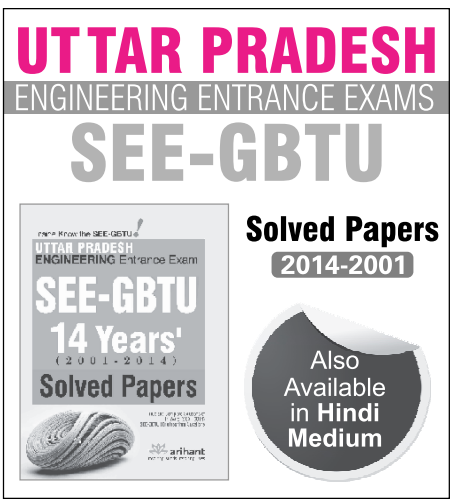
Let f be a real function, continuous on the closed interval $[a, b]$ and differentiable in the open interval (a, b). Then, there is atleast one point c in the open interval (a, b), such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



Geometrically Any chord of the curve $y = f(x)$, there is a point on the graph, where the tangent is parallel to this chord.

➦ In the particular case, where $f(a) = f(b)$, the expression $\frac{f(b)-f(a)}{b-a}$ becomes zero. Thus, when $f(a) = f(b)$, $f'(c) = 0$ for some c in (a, b). Thus, Rolle's theorem becomes a particular case of the mean value theorem.



Mathematics SPECTRUM for You

In this section, we invite our readers to send their problems, we will publish the best explanation of them.

Q1 **Sir, I usually get problems in finding the equation of asymptotes. Please explain it.**

(Neha Singh, Meerut)

➥ Dear Neha, first of all you have to aware of concept of asymptotes.

A straight line is said to be an asymptote of a curve $y = f(x)$, if the perpendicular distance of the point $P(x, y)$ on the curve from the line tends to zero when x and y or both tend to infinity. An asymptote parallel to Y-axis may be referred to as a vertical asymptote and that parallel to X-axis as a horizontal asymptote. An asymptote which is not parallel to either axis may be described as an oblique asymptote.

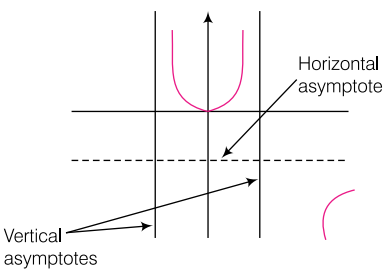
(a) **Vertical asymptotes** A line $x = a$ is a vertical asymptote of the graph $y = f(x)$, if $\lim\limits_{x \to a^+} f(x) = \pm \infty$ or $\lim\limits_{x \to a^-} f(x) = \pm \infty$.

For polynomial Suppose that f is a rational function $f(x) = \dfrac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomial functions of x. Then, the line $x = a$ is a vertical asymptote of the graph of $y = f(x)$, if $Q(a) = 0$ but $P(a) \neq 0$.

Example Find the vertical asymptotes of $f(x) = \dfrac{x^2}{4 - x^2}$.

Sol. Let $f(x) = \dfrac{P(x)}{Q(x)}$

Here, $P(x) = x^2$



and $Q(x) = 4 - x^2 = (2 - x)(2 + x)$

Now, $Q(x) = 0$

$\Rightarrow \quad (2 - x)(2 + x) = 0$

$\Rightarrow \quad x = -2$ and $x = 2$

Also, $P(-2)$ and $P(2) \neq 0$. So, $x = 2$ and $x = -2$ are vertical asymptotes.

(b) **Horizontal asymptotes** A line $y = b$ is a horizontal asymptote of the graph $y = f(x)$, if $\lim\limits_{x \to \infty} f(x) = b$ or $\lim\limits_{x \to \infty} f(x) = -b$.

Example Find the horizontal asymptotes of $f(x) = \dfrac{x^2}{4 - x^2}$.

Sol. $\lim\limits_{x \to \infty} \dfrac{x^2}{4 - x^2} = \lim\limits_{x \to \infty} \dfrac{1}{\frac{4}{x^2} - 1} = -1$

So, $y = -1$ is a horizontal asymptote.

(c) **Oblique asymptotes** A line $y = mx + c$ is an oblique asymptote of the curve $y = f(x)$, if

$$\lim_{x \to \pm\infty} \frac{y}{x} = \lim_{x \to \pm\infty} \frac{f(x)}{x} = m$$

and $$\lim_{x \to \pm\infty} (y - mx) = \lim_{x \to \pm\infty} [f(x) - mx] = c$$

[Q2] **I am facing problems in solving questions based on conditional probability. Please clear my doubt by taking example.**

(Sudhir Rawat, Mumbai)

➥ Dear Sudhir, first of all you have to aware of concept of conditional probability.

The probability of occurrence of an event A, given that B has already occurred is called the conditional probability of occurrence of A. It is denoted by $P\left(\frac{A}{B}\right)$.

If the event B has already occurred, then the sample space reduces to B. Now, the outcomes favourable to the occurrence of A (given that B has already occurred) are those that are common to both A and B, that is those which belong to $A \cap B$.

Thus, $$P\left(\frac{A}{B}\right) = \frac{N_{A \cap B}}{N_B}$$

where, $N_{A \cap B}$ is the number of elements in $A \cap B$ and $N_B \neq 0$, the number of elements in B and N is the total number of elements in S.

$$\Rightarrow \quad P\left(\frac{A}{B}\right) = \frac{N_{A \cap B} / N}{N_B / N} = \frac{P(A \cap B)}{P(B)}$$

Hence, $$P(A \cap B) = \begin{cases} P(B)\, P\left(\frac{A}{B}\right) & \text{if } P(B) \neq 0 \\ P(A)\, P\left(\frac{B}{A}\right) & \text{if } P(A) \neq 0 \end{cases}$$

Example A die is thrown twice and the sum of the numbers appearing is observed to be 7. What is the conditional probability that the number 2 has appeared atleast once?

Sol. A die is thrown twice. Therefore, $n(S) = 6 \times 6 = 36$

Event E = The sum of the numbers appearing is 7.

$\therefore \quad E = \{(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4,3)\}$

$\Rightarrow \quad n(E) = 6$

Event F = The number 2 has appeared atleast once.

$\therefore F = \{(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (1, 2), (3, 2), (4, 2), (5, 2), (6, 2)\}$

$\Rightarrow \quad n(F) = 11$

Now, $E \cap F = \{(2, 5), (5, 2)\}$

$\Rightarrow \quad P(E \cap F) = \frac{2}{36}$

$\therefore$ Required probability $= P\left(\frac{F}{E}\right) = \frac{P(E \cap F)}{P(E)}$

$$= \frac{\frac{2}{36}}{\frac{6}{36}} = \frac{1}{3}$$

[Q3] **Sir, I am confused in solving the problems based on reduction formulae. Please explain by taking an example.** **(Mayank, Meerut)**

➥ Dear Mayank, first of all you have to aware of concept of reduction formula.

Any formula which express an integral in terms of another integral which is simpler and is of the same class as the first, is called a reduction formula for the first integral.

The successive application of the reduction formula enables us to express integrals of a particular class in term of the simplest integral of the class.

Example Evaluate $\int \text{cosec}^n x\,dx$.

Sol. Let $I_n = \int \text{cosec}^n x\,dx = \int \text{cosec}^2 x\, \text{cosec}^{(n-2)} x\,dx$

$\Rightarrow I_n = -\cot x\, \text{cosec}^{(n-2)} x + \int (\cot x)(n-2)[-\text{cosec}^{(n-3)} x \text{cosec}\, x \cot x]dx$

$\Rightarrow I_n = -\cot x \text{cosec}^{(n-2)} x - (n-2)\int (\text{cosec}^2 x - 1)\, \text{cosec}^{(n-2)} x\,dx$

$\Rightarrow I_n = -\cot x \text{cosec}^{(n-2)} x - (n-2)\int \text{cosec}^n x\,dx + (n-2)\int \text{cosec}^{(n-2)} x\,dx$

$\Rightarrow I_n = -\cot x \text{cosec}^{(n-2)} x - (n-2) I_n + (n-2) I_{(n-2)}$

$\Rightarrow I_n + (n-2) I_n = -\cot x \text{cosec}^{(n-2)} x + (n-2) I_{(n-2)}$

$\Rightarrow I_n (1 + n - 2) = -\cot x \text{cosec}^{(n-2)} x + (n-2) I_{(n-2)}$

$\Rightarrow I_n (n-1) = -\cot x \text{cosec}^{(n-2)} x + (n-2) I_{(n-2)}$

$$\Rightarrow I_n = \frac{-\cot x \text{cosec}^{(n-2)} x}{(n-1)} + \left(\frac{n-2}{n-1}\right) I_{(n-2)}$$

$$\Rightarrow I_n = \frac{\cot x \text{cosec}^{(n-2)} x}{-(n-1)} + I_{(n-2)}$$

Some Important Results

We can use directly some important results.

(i) $I_n = \int \sin^n x\,dx$ (where, $n \in I^+$ $= \frac{-\cos x \sin^{(n-1)} x}{n} + \left(\frac{n-1}{n}\right) I_{(n-2)}$

(ii) $I_n = \int \cos^n x\,dx$ (where, $n \in I^+$) $= \frac{\sin x \cos^{(n-1)} x}{n} + \left(\frac{n-1}{n}\right) I_{(n-2)}$

(iii) $I_n = \int \tan^n x\,dx$ (where, $n \in I^+$) $= \frac{\tan^{(n-1)} x}{(n-1)} - I_{(n-2)}$

(iv) $I_n = \int \cot^n x\,dx$ (where, $n \in I^+$) $= \frac{-\cot^{(n-1)} x}{(n-1)} - I_{(n-2)}$

(v) $I_n = \int \sec^n x\,dx$ (where, $n \in I^+$) $= \frac{\tan x \sec^{(n-2)} x}{n-1} + \left(\frac{n-2}{n-1}\right) I_{(n-2)}$

(vi) $I_{m,n} = \int \sin^m x \cos^n x\,dx$ (where, $m, n \in I^+$)

$$= \frac{\sin^{(m+1)} x \cos^{(n-1)} x}{(m+n)} + \left(\frac{n-1}{m+n}\right) I_{(m, n-2)}$$

(vii) $I_{m,n} = \int \cos^m x \cos nx\,dx$ (where, $m, n \in I^+$)

$$= \frac{n \sin nx \cos x - m \cos nx \sin x}{n^2 - m^2} \cdot \cos^{(m-1)} x - \frac{m(m-1)}{(n^2 - m^2)} I_{(m-2, n)}$$

Q4 Sir, In which type of problems in limits we will use L'Hospital's rule. (Riya Sharma, Delhi)

Dear Riya, we will use L'hospital's rule in case of indeterminate form $\left(\frac{0}{0}, \frac{\infty}{\infty}\right)$. If $f(x)$ and $g(x)$ are two functions of x such that

(i) $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$

(ii) both are continuous at $x = a$

(iii) both are differentiable at $x = a$

(iv) $f'(x)$ and $g'(x)$ are continuous at the point $x = a$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ provided that $g(a) \neq 0$.

Above rule is also applicable, if $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$

If $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ assumes the indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and $f'(x) \cdot g'(x)$ satisfy all the conditions embeded in L' hospital's rule. We can repeat the application of this rule on $\frac{f'(x)}{g'(x)}$ to get

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} \frac{f''(x)}{g''(x)}$$

When we solving the such types of problems we follow the steps below:

***Step* I** Whenever, we solve the limit based problem, we see that the problem form indeterminate form or not.

***Step* II** If any form is given, then we check the form $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$.

***Step* III** If the form either both of these form, we apply L' hospital's rule i.e. differentiate the function numerator and denominator separately until the forms $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$ resolve indeterminate form.

***Step* IV** Finally, we apply the limit and get the desired result.

Example Find $\lim_{x \to 0} \frac{\log \log(1 - x^2)}{\log \log \cos x}$.

Sol. Here, $\lim_{x \to 0} \frac{\log \log(1 - x^2)}{\log \log \cos x}$ is of the form $\left(\frac{\infty}{\infty}\right)$.

$$\lim_{x \to 0} \frac{\log\log(1 - x^2)}{\log\log\cos x} = \lim_{x \to 0} \frac{\frac{1}{\log(1 - x^2)} \cdot \frac{1}{(1 - x^2)}(-2x)}{\frac{1}{\log \cos x} \cdot \frac{1}{\cos x}(-\sin x)}$$

[differentiating numerator and denominator separately]

$$= 2 \lim_{x \to 0} \frac{x \cos x \log \cos x}{\sin x \cdot (1 - x^2) \log(1 - x^2)}$$

$$= 2 \lim_{x \to 0} \frac{x}{\sin x} \cdot \lim_{x \to 0} \frac{\cos x}{1 - x^2} \cdot \lim_{x \to 0} \frac{\log \cos x}{\log(1 - x^2)}$$

$$= 2 \times 1 \times 1 \times \lim_{x \to 0} \frac{\log \cos x}{\log(1 - x^2)} \qquad \left[\text{form } \frac{0}{0}\right]$$

$$= 2 \lim_{x \to 0} \frac{\frac{1}{\cos x} \cdot (-\sin x)}{\frac{1}{1 - x^2} \cdot (-2x)}$$

[differentiating numerator and denominator separately]

$$= 2 \times \frac{1}{2} \cdot \lim_{x \to 0} \left(\frac{\sin x}{x} \cdot \frac{1 - x^2}{\cos x}\right) = 1$$

BRAIN Teasers

To boost up your ranks in JEE Main and Advanced

1. If a real-valued function f is defined as $f(x) = \lim\limits_{\alpha \to 0} \dfrac{\begin{bmatrix}\cos(x+4\alpha) - 4\cos(x+3\alpha) \\ + 6\cos(x+2\alpha) - 4\cos(x+\alpha) + \cos x\end{bmatrix}}{\alpha^4}$, then find the value of $|f_{min} - f_{max}|$.

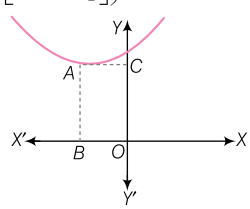
2. If the quadratic polynomial $f(x) = kx^2 - (2k+3)x + 6, \forall x \neq 0$ is always positive for exactly three different negative integral values of x, then find the set of values of k.

3. Find the number of ordered pairs (α, β), which satisfies the system of equations

$\cot^2(\alpha - \beta) - (1+\sqrt{3})\cot(\alpha - \beta) + \sqrt{3} = 0$ and $\cos\beta = \dfrac{\sqrt{3}}{2}$, $\forall\ 0 < \alpha, \beta < \pi$.

4. Find the minimum value of the function $f(x) = \dfrac{12}{\pi}\left(\text{cosec}^{-1}\left[3x^2 + \dfrac{5}{4}\right] + \sec^{-1}\left[3x^2 + \dfrac{1}{4}\right]\right)$, where $[\cdot]$ denotes the greatest integer function.

5. If graph of a quadratic polynomial $f(x) = ax^2 + bx + c$ is shown below in the figure in which $AB = 2$, $AC = 3$ and $b^2 - 4ac = -4$, then find the range of the function $H(x) = \left(a + \dfrac{1}{2}\right)x^2 + (b+2)x - \left(c - \dfrac{1}{2}\right)$, $\forall\ x \in [-4, 0]$.



6. If a function $f : [0, \infty) \to R$ is continuous and strictly increasing, defined as $f^3(x) = \int_0^x tf^2(t)\,dt$ and a normal drawn to $y = f(x)$, where slope is $-\dfrac{1}{2}$, then find the length of the intercept made by this normal on Y-axis.

7. If $f(x) = x^3 + \log_e x + 1$ is a real-valued function of x, whose inverse is $g(x)$, then determine the value of $g''(2)$ or $\left(\dfrac{d^2 f^{-1}}{dx^2}\right)_{x=2}$.

8. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three vectors in a plane such that $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = |\mathbf{a} + \mathbf{b}| = 1$, $\mathbf{c}\cdot\mathbf{a} = 0$, $\mathbf{b}\cdot\mathbf{c} > 0$. If a vector $\mathbf{p} = x\mathbf{a} + y\mathbf{c}$ satisfies the condition $0 \le \mathbf{p}\cdot\mathbf{a} \le 1$, $0 \le \mathbf{p}\cdot\mathbf{b} \le 1$, then find the maximum value of $\mathbf{p}\cdot\mathbf{c}$.

9. Find the equation of the curve passing through the origin, if the middle point of the segment of its normal from any point of the curve to X-axis lies on the parabola $2y^2 = x$.

10. A tosses 2 fair coins and B tosses 3 fair coins. The game is won by the person who throws greater number of heads. In case of tie, the game is continued under identical rules until someone finally wins the game. Find the probability that A finally wins the game.

Answer *with* Explanations

1. $f(x) = \lim_{\alpha \to 0} \dfrac{\left[\begin{array}{r}\{\cos(x+4\alpha)+\cos x\} - 4\{\cos(x+3\alpha)\\ +\cos(x+\alpha)\} + 6\cos(x+2\alpha)\end{array}\right]}{\alpha^4}$

$= \lim_{\alpha \to 0} \dfrac{\left[\begin{array}{r}2\cos(x+2\alpha)\cos 2\alpha - 8\cos(x+2\alpha)\\ \cos\alpha + 6\cos(x+2\alpha)\end{array}\right]}{\alpha^4}$

$= \lim_{\alpha \to 0} \dfrac{2\cos(x+2\alpha)(\cos 2\alpha - 4\cos\alpha + 3)}{\alpha^4}$

$= \lim_{\alpha \to 0} \dfrac{2\cos x(2\cos^2\alpha - 4\cos\alpha + 2)}{\alpha^4}$

$= \lim_{\alpha \to 0} 4\cos x \cdot \dfrac{(1-\cos\alpha)^2}{\alpha^4} = \cos x$

So, maximum value of $f(x) = 1$ and minimum value of $f(x) = -1$.

$\therefore$ Difference between maximum and minimum values $= 2$

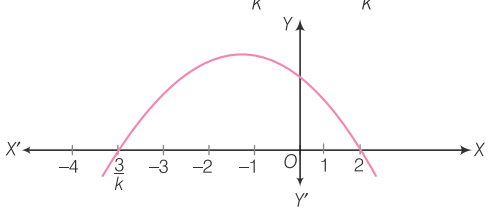
$f(x) = kx^2 - (2k+3)x + 6, \forall \neq 0$ is always positive for exactly three different negative integral values of x.

2. $f(x) = kx^2 - (3+2k)x + 6 = (kx-3)(x-2)$

Here, roots of the equation $f(x) = 0$ are 2 and $\dfrac{3}{k}$ and $f(0) = 6$.

Since, $f(x)$ should be positive for exactly three negative integral values of x.

Therefore, graph of $f(x)$ must be a downward parabola passing through $x = 2$ and $x = \dfrac{3}{k}$ and $-4 \le \dfrac{3}{k} < -3$.



$\therefore \quad k \in \left(-1, -\dfrac{3}{4}\right]$

3. We have, $\cos\beta = \dfrac{\sqrt{3}}{2} \Rightarrow \beta = \dfrac{\pi}{6}$

Let $\cot(\alpha - \beta) = t$

$\Rightarrow t^2 - t - \sqrt{3}t + \sqrt{3} = 0 \Rightarrow t(t-1) - \sqrt{3}(t-1) = 0$

$\Rightarrow (t-1)(t-\sqrt{3}) = 0$

$\Rightarrow \cot(\alpha - \beta) = 1$ or $\cot(\alpha - \beta) = \sqrt{3}$

$\because \quad \alpha \in (0, \pi)$

$\therefore \quad \alpha - \beta = \dfrac{\pi}{6} \Rightarrow \alpha = \dfrac{\pi}{6} + \dfrac{\pi}{6} = \dfrac{\pi}{3}$

$\Rightarrow (\alpha, \beta) = \left(\dfrac{\pi}{3}, \dfrac{\pi}{6}\right)$

and $\alpha - \beta = \dfrac{\pi}{4} \Rightarrow \alpha = \dfrac{\pi}{4} + \dfrac{\pi}{6} = \dfrac{5\pi}{12}$

$\Rightarrow (\alpha, \beta) = \left(\dfrac{5\pi}{12}, \dfrac{\pi}{6}\right)$

So, the possible ordered pairs of (α, β) are $\left(\dfrac{\pi}{3}, \dfrac{\pi}{6}\right), \left(\dfrac{5\pi}{12}, \dfrac{\pi}{6}\right)$.

Hence, 2 ordered pairs exist.

4. Let $M = \operatorname{cosec}^{-1}\left[3x^2 + 1 + \dfrac{1}{4}\right] + \sec^{-1}\left[3x^2 + \dfrac{1}{4}\right]$

$= \operatorname{cosec}^{-1}\left[1 + \left(3x^2 + \dfrac{1}{4}\right)\right] + \sec^{-1}\left[3x^2 + \dfrac{1}{4}\right]$

where, $\left[3x^2 + \dfrac{1}{4}\right] \ge 1$

Now, M will be minimum, when $\left[3x^2 + \dfrac{1}{4}\right]$ is minimum.

$\Rightarrow \quad \left[3x^2 + \dfrac{1}{4}\right] = 1$

$\therefore \quad M_{\min} = \operatorname{cosec}^{-1} 2 + \sec^{-1} 1 = \dfrac{\pi}{6}$

Hence, $\dfrac{12}{\pi} M = \dfrac{12}{\pi} \times \dfrac{\pi}{6} = 2$

5. By graph it is clear that $\dfrac{-b}{2a} = -3, \dfrac{-D}{4a} = 2$. Also $D = 2$

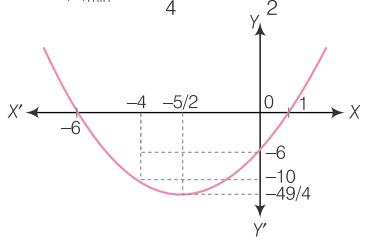
By simplifying above, we get $a = \dfrac{1}{2}, b = 3$ and $c = \dfrac{13}{2}$

Hence, $H(x) = \left(a + \dfrac{1}{2}\right)x^2 + (b+2)x - \left(c - \dfrac{1}{2}\right)$

$H(x) = x^2 + 5x - 6 = (x+6)(x-1)$

$\Rightarrow \quad H(-4) = -10$ and $H(0) = -6$

$\therefore \quad H(x)_{\min} = -\dfrac{49}{4}$ at $x = -\dfrac{5}{2}$



So, range of $H(x)$ is $\left[-\dfrac{49}{4}, -6\right]$.

6. Given, $f^3(x) = \int_0^x t\, f^2(t)\, dt$, where $y = f(x)$ ...(i)

On differentiating both sides of Eq. (i) by using Leibnitz rule, we get

$3f^2(x)f'(x) = xf^2(x)$

As $f(x)$ is strictly increasing, so $f(x) \neq 0$.

$\therefore \quad 3f'(x) = x \Rightarrow f'(x) = \dfrac{x}{3}$

On integrating both sides w.r.t. x, we get

$f(x) = \dfrac{x^2}{6} + C$

But $f(0) = 0 \Rightarrow C = 0$

$\therefore \quad f(x) = \dfrac{x^2}{6}, x \ge 0$

Slope of the tangent $P(x_1, y_1)$ to the curve $f(x) = \dfrac{x_1}{3}$

$\therefore$ Slope of normal $= \dfrac{-3}{x_1}$

i.e. $\frac{-1}{2} = \frac{-3}{x_1}$ [given]

$\Rightarrow$ $x_1 = 6$ and $y_1 = 6$ $\left[\because y_1 = f(6) = \frac{6^2}{6} = 6\right]$

So, equation of normal is $(y - 6) = \frac{-1}{2}(x - 6)$.

$\therefore$ For y-intercept, put $x = 0$, we get $y = 9$.

7. Given, $f(x) = x^3 + \log_e x + 1$ as $g(x)$ is the inverse of f, then $g\{f(x)\} = x$

$\Rightarrow$ $g'\{f(x)\} \cdot f'(x) = 1 \Rightarrow g'\{f(x)\} = \frac{1}{f'(x)}$

$\Rightarrow$ $g''\{f(x)\} = -\frac{f''(x)}{\{f'(x)^3\}}$

When $x = 1$, $g''\{f(1)\} = g''(2) = -\frac{f''(1)}{\{f'(1)^3\}}$

So, $f'(x) = 3x^2 + \frac{1}{x} \Rightarrow f''(x) = -\frac{1}{x^2} + 6x$

At $x = 1$, $g''(2) = -\frac{5}{64}$

8. $\mathbf{p} = x\mathbf{a} + y\mathbf{c}$ when $0 \le \mathbf{p}\cdot\mathbf{a} \le 1$ and $0 \le \mathbf{p}\cdot\mathbf{b} \le 1$ with $\mathbf{a}$ gives $\mathbf{p}\cdot\mathbf{a} = x \Rightarrow 0 \le x \le 1$.

$$\mathbf{p}\cdot\mathbf{b} = x\mathbf{a}\cdot\mathbf{b} + y\mathbf{c}\cdot\mathbf{b} = -\frac{x}{2} + \frac{\sqrt{3}}{2}y = \frac{\sqrt{3}y - x}{2}$$

$$0 \le \frac{\sqrt{3}y - x}{2} \le 1 \Rightarrow 0 \le \sqrt{3}y - x \le 2$$

$\Rightarrow$ $x \le \sqrt{3}y \le 2 + x$

$\Rightarrow$ $\frac{x}{\sqrt{3}} \le y \le \frac{2 + x}{\sqrt{3}}$

So, $y_{max} = \frac{2 + 1}{\sqrt{3}} = \sqrt{3}$

Also, $\mathbf{p}\cdot\mathbf{c} = y$

Hence, $(\mathbf{p}\cdot\mathbf{c})$ is maximum when y is maximum, thus $(\mathbf{p}\cdot\mathbf{c})_{max} = \sqrt{3}$.

9. Equation of normal to curve at point $P(x_1, y_1)$ is $y - y_1 = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}(x - x_1)$ and it intersect X-axis, where $y = 0$.

i.e. $x = y_1\left(\frac{dy}{dx}\right)_{(x_1, y_1)} + x_1$

$\Rightarrow$ $Q\left(\frac{dy}{dx}\cdot y_1 + x_1, 0\right)$

Since, middle point (R) of segment PQ lies on the curve $2y^2 = x$.

So, $R \equiv \left[\frac{x_1 + \left(\frac{dy}{dx}\right)_{(x_1, y_1)} y_1 + x_1}{2}, \frac{y_1}{2}\right]$ lies on the curve $2y^2 = x$.

$\Rightarrow$ $2\left(\frac{y_1}{2}\right)^2 = \frac{2x_1 + \left(\frac{dy}{dx}\right)_{(x_1, y_1)} y_1}{2}$

$\Rightarrow$ $y_1^2 = 2x_1 + \left(\frac{dy}{dx}\right)_{(x_1, y_1)} y_1$

$\Rightarrow$ $\left(\frac{dy}{dx}\right)_{(x_1, y_1)} \cdot y_1 = y_1^2 - 2x_1$

$\Rightarrow$ $\left(\frac{dy}{dx}\right)y = y^2 - 2x$ ...(i)

[replacing (x_1, y_1) by (x, y)]

Substituting $y^2 = t \Rightarrow 2y\frac{dy}{dx} = \frac{dt}{dx}$

$\Rightarrow$ $y\frac{dy}{dx} = \frac{1}{2}\cdot\frac{dt}{dx}$ ...(ii)

From Eqs. (i) and (ii), we get $\frac{1}{2}\cdot\frac{dt}{dx} = t - 2x$

$\Rightarrow \frac{dt}{dx} - 2t + 4x = 0$, which is linear differential equation with

$$\text{IF} = e^{\int -2dx} = e^{-2x}$$

$\therefore$ Solution of differential equation is given by

$t\cdot e^{-2x} = \int(-4x)e^{-2x}dx + C$

$\Rightarrow$ $t\cdot e^{-2x} = -4\left[\frac{x\cdot e^{-2x}}{-2} - \int\frac{1\cdot e^{-2x}}{-2}dx\right]$

$\Rightarrow$ $t\cdot e^{-2x} = 2xe^{-2x} + e^{-2x} + C$

$\Rightarrow$ $y^2 = 2x + 1 + Ce^{2x}$

Since, the curve passes through origin.

$\Rightarrow$ $C = -1$

$\therefore$ $y^2 = 2x - e^{2x} + 1$ is the required equation of curve.

10. For a particular game, let

A_i = number of heads obtained by A when he tosses two coins and B_i = number of heads obtained by B when he tosses three games.

If $E = \{A$ wins a particular game$\}$, then

$P(E) = P(A_1 \cap B_0$ or $A_2 \cap B_0$ or $A_2 \cap B_1)$

$= P(A_1 \cap B_0) + P(A_2 \cap B_0) + P(A_2 \cap B_1)$

$= P(A_1)P(B_0) + P(A_2)P(B_0) + P(A_2)P(B_1)$

[independent events]

$$= {}^2C_1\left(\frac{1}{2}\right)\cdot\left(\frac{1}{2}\right)\cdot\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2\cdot\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2\cdot{}^3C_1\left(\frac{1}{2}\right)\cdot\left(\frac{1}{2}\right)^2$$

$$= \frac{2}{32} + \frac{1}{32} + \frac{3}{32} = \frac{3}{16}$$

If $F = \{A$ and B tie a particular game$\}$, then

$P(F) = P(A_0 \cap B_0$ or $A_1 \cap B_1$ or $A_2 \cap B_2)$

$= P(A_0)P(B_0) + P(A_1)P(B_1) + P(A_2)P(B_2)$

$$= \left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 + {}^2C_1\left(\frac{1}{2}\right)\cdot\left(\frac{1}{2}\right)\cdot{}^3C_1\left(\frac{1}{2}\right)\cdot\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\cdot{}^3C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)$$

$$= \frac{1}{32} + \frac{6}{32} + \frac{3}{32} = \frac{5}{16}$$

Hence, $P\{A$ finally wins the game$\}$

$= P\{E$ or FE or FFE or ...$\}$

$= P(E) + P(F)P(E) + \{P(F)\}^2P(E) + \ldots$

$$= \frac{P(E)}{1 - P(F)} = \frac{3/16}{1 - 5/16} = \frac{3}{11}$$

TEST RIDER

The Simulator test series towards JEE Main and Advanced

JEE MAIN RIDE 1

Instructions *This question paper contains 30 questions each having four options, out of which only one option is correct choose the correct option.*

1. Let a, b and c be real numbers and $a \neq 0$. If α is a root of $a^2x^2 + bx + c = 0$, β is a root of $a^2x^2 - bx - c = 0$ and $0 < \alpha < \beta$, then the equation $a^2x^2 + 2bx + 2c = 0$ has a root γ that always satisfies

(a) $\gamma = \frac{\alpha + \beta}{2}$ (b) $\gamma = \alpha + \frac{\beta}{2}$

(c) $\gamma = \alpha$ (d) $\alpha < \gamma < \beta$

2. If $A_0\ A_1\ A_2\ A_3\ A_4\ A_5$ regular hexagon inscribed in unit radius circle, then multiplication of lengths of line segments A_0A_1, A_0A_2 and A_0A_4 is

(a) $\frac{3}{4}$ (b) $3\sqrt{3}$ (c) 3 (d) $\frac{3\sqrt{3}}{2}$

3. If $a_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$, then $\sum_{r=0}^{n} \frac{r}{{}^nC_r}$ is equal to

(a) $(n-1)a_n$ (b) na_n

(c) $\frac{1}{2}na_n$ (d) None of these

4. A function and from the set of natural numbers to integers defined by $f(n) = \begin{cases} \frac{n-1}{2}, & \text{when } n \text{ is odd} \\ \frac{-n}{2}, & \text{when } n \text{ is even} \end{cases}$ is

(a) one-one but not onto (b) onto but not one-one

(c) both one-one and onto (d) neither one-one nor onto

5. If $E(\theta) = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$ and θ, ϕ differ by an odd multiple of $\frac{\pi}{2}$, then $E(\theta)E(\phi)$ is a

(a) null matrix (b) unit matrix

(c) diagonal matrix (d) None of these

6. If the system of linear equations

$$x + 2ay + az = 0$$

$$x + 3by + bz = 0$$

and

$$x + 4cy + cz = 0$$

has a non-zero solution, then a, b and c

(a) are in AP (b) are in GP

(c) are in HP (d) satisfy $a + 2b + 3c = 0$

7. A student is allowed to select atmost n books from a collection of $(2n+1)$ books. If the total number of ways in which he can select books is 63, then n is equal to

(a) 4 (b) 3 (c) 7 (d) 8

8. In a circle of radius r, an isosceles ΔABC is inscribed with $AB = AC$. If the ΔABC has perimeter $P = 2[\sqrt{2hr - h^2} + \sqrt{2hr}]$ and area $A = h\sqrt{2hr - h^2}$, where h is the altitude from A to BC, then $\lim_{h \to 0^+} \frac{A}{P^3}$ is equal to

(a) $128r$ (b) $\frac{1}{128r}$

(c) $\frac{1}{64r}$ (d) None of these

9. The trigonometric equation $\sin^{-1}x = 2\sin^{-1}a$ has a solution for

(a) $\frac{-1}{\sqrt{2}} < a < \frac{1}{\sqrt{2}}$ (b) all real values of a

(c) $|a| < \frac{1}{2}$ (d) $|a| \geq \frac{1}{\sqrt{2}}$

10. If the equation of the locus of a point equidistant from the points (a_1, b_1) and (a_2, b_2) is $(a_1 - a_2)x + (b_1 - b_2)y + c = 0$, then the value of c is

(a) $\frac{1}{2}(a_2^2 + b_2^2 - a_1^2 - b_1^2)$ (b) $a_1^2 - a_2^2 + b_1^2 - b_2^2$

(c) $\frac{1}{2}(a_1^2 + a_2^2 + b_1^2 + b_2^2)$ (d) $\sqrt{a_1^2 + b_1^2 - a_2^2 - b_2^2}$

11. If the lines $3x-4y-7=0$ and $2x-3y-5=0$ are two diameters of a circle of area 49π sq units, then the equation of the circle is
(a) $x^2+y^2+2x-2y-62=0$
(b) $x^2+y^2-2x+2y-62=0$
(c) $x^2+y^2-2x+2y-47=0$
(d) $x^2+y^2+2x-2y-47=0$

12. The values of constants a and b so as to make the function $f(x)=\begin{cases}\dfrac{1}{|x|}, & |x|\geq 1\\ ax^2+b, & |x|<1\end{cases}$ continuous as well as differentiable for all x, are
(a) $a=\dfrac{-1}{2}, b=\dfrac{3}{2}$ (b) $a=\dfrac{1}{2}, b=\dfrac{3}{2}$
(c) $a=\dfrac{-1}{2}, b=\dfrac{-3}{2}$ (d) None of these

13. Consider
Statement I $(p\wedge\sim q)\wedge(\sim p\wedge q)$ is a fallacy.
Statement II $(p\to q)\leftrightarrow(\sim q\to\sim p)$ is a tautology.
(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

14. The range of values of a for which the function $f(x)=x^3+(a+2)x^2+3ax+5$ may be monotonic in R, is
(a) $a<1$ (b) $1<a<4$
(c) $a>4$ (d) None of these

15. a, b and c are three distinct real numbers and they are in a GP. If $a+b+c=xb$, then
(a) $x<-1$ or $x>3$ (b) $x<-3$ or $x>2$
(c) $x<-4$ or $x>3$ (d) None of these

16. The value of $\sqrt{2}\int\dfrac{\sin x\,dx}{\sin\left(x-\dfrac{\pi}{4}\right)}$ is
(a) $x+\log\left|\cos\left(x-\dfrac{\pi}{4}\right)\right|+C$ (b) $x-\log\left|\sin\left(x-\dfrac{\pi}{4}\right)\right|+C$
(c) $x+\log\left|\sin\left(x-\dfrac{\pi}{4}\right)\right|+C$ (d) $x-\log\left|\cos\left(x-\dfrac{\pi}{4}\right)\right|+C$

17. If $I=\int_0^1\dfrac{\sin x}{\sqrt{x}}dx$ and $J=\int_0^1\dfrac{\cos x}{\sqrt{x}}dx$, then which one of the following is true?
(a) $I>\dfrac{2}{3}$ and $J>2$ (b) $I<\dfrac{2}{3}$ and $J<2$
(c) $I<\dfrac{2}{3}$ and $J>2$ (d) $I>\dfrac{2}{3}$ and $J<2$

18. Three natural numbers are taken at random from the set $A=\{x:1\leq x\leq 100, x\in N\}$. The probability that the AM of the numbers taken is 25, is
(a) $\dfrac{{}^{77}C_2}{{}^{100}C_3}$ (b) $\dfrac{{}^{25}C_2}{{}^{100}C_3}$ (c) $\dfrac{{}^{74}C_2}{{}^{100}C_3}$ (d) None of these

19. The number of points with integral coordinates that lie in the interior of the region common to the circle $x^2+y^2=16$ and the parabola $y^2=4x$, is
(a) 8 (b) 10
(c) 16 (d) None of these

20. Let R be the real line. Consider the following subsets of the plane $R\times R$, $S=\{(x,y):y=x+1 \text{ and } 0<x<2\}$
$T=\{(x,y):x-y \text{ is an integer}\}$
Which one of the following is true?
(a) Neither S nor T is an equivalence relation on R
(b) Both S and T are equivalence relation on R
(c) S is an equivalence relation on R but T is not
(d) T is an equivalence relation on R but S is not

21. The vector $\mathbf{a}=\alpha\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\beta\hat{\mathbf{k}}$ lies in the plane of the vectors $\mathbf{b}=\hat{\mathbf{i}}+\hat{\mathbf{j}}$ and $\mathbf{c}=\hat{\mathbf{j}}+\hat{\mathbf{k}}$ and bisects the angle between $\mathbf{b}$ and $\mathbf{c}$. Then, which one of the following gives possible values of α and β?
(a) $\alpha=2,\beta=2$ (b) $\alpha=1,\beta=2$
(c) $\alpha=2,\beta=1$ (d) $\alpha=1,\beta=1$

22. If n is the number of solutions of the equation $|\cot x|=\cot x+\dfrac{1}{\sin x}\ (0<x<2\pi)$, then n is equal to
(a) 1 (b) 2 (c) 3 (d) 4

23. If the mean deviation of number $1, 1+d, 1+2d, \cdots, 1+100d$ from their mean is 255, then d is equal to
(a) 10.0 (b) 20.0 (c) 10.1 (d) 20.2

24. **Statement I** If P is chosen at random in the closed interval $[0,5]$, then the probability that the equation $x^2+Px+\dfrac{1}{4}(P+2)=0$ has real roots, is $\dfrac{3}{5}$.
Statement II If discriminant ≥ 0, then roots of the quadratic equation are always real.
(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true ; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

25. ABC is an equilateral triangular plot. An electric pole stands at the vertex and makes an angle of 60° at either of the other two vertices. If the height of the triangle is 100 m, then height of the pole is
(a) 200 m (b) 100 m
(c) 150 m (d) None of these

26. In a ΔABC, if $a^2+b^2=101c^2$, then the value of $\dfrac{\cot C}{\cot A+\cot B}$ is
(a) 50 (b) 45
(c) 40 (d) None of these

27. If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are non-coplanar vectors and p, q are real numbers, then the equality $[3\,\mathbf{u}\ p\mathbf{v}\ p\mathbf{w}] - [p\mathbf{v}\ \mathbf{w}\ q\mathbf{u}] + [2\mathbf{w}\ q\mathbf{v}\ q\mathbf{u}] = 0$ holds for
(a) exactly two values of (p,q)
(b) more than two but not all values of (p, q)
(c) all values of (p,q)
(d) exactly one value of (p,q)

28. Let $f:[-1,2] \to [0,\infty)$ be a continuous function such that $f(x) = f(1-x)$ for all $x \in [-1,2]$. If $R_1 = \int_{-1}^{2} x\, f(x)\, dx$ and R_2 are the area of the region bounded by $y = f(x)$, $x = -1$, $x = 2$ and X-axis, then
(a) $R_1 = 2R_2$
(b) $R_1 = 3R_2$
(c) $2R_1 = R_2$
(d) $3R_1 = R_2$

29. The normal at point P on the ellipse $x^2 + 4y^2 = 16$ meets the X-axis at Q. If M is the mid-point of the line segment PQ, then the locus of M intersects the latusrectum of the given ellipse at the points
(a) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\frac{2}{7}\right)$ (b) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\frac{\sqrt{19}}{4}\right)$
(c) $\left(\pm 2\sqrt{3}, \pm\frac{1}{7}\right)$ (d) $\left(\pm 2\sqrt{3}, \pm\frac{4\sqrt{3}}{7}\right)$

30. If $y(t)$ is a solution of $(1+t)\frac{dy}{dt} - ty = 1$ and $y(0) = -1$, then $y(1)$ is equal to
(a) $\frac{-1}{2}$ (b) $e + \frac{1}{2}$
(c) $e - \frac{1}{2}$ (d) $\frac{1}{2}$

Answer *with* Explanations

1. ***(d)*** Let $f(x) = a^2x^2 + 2bx + 2c = 0$

Given, $a^2\alpha^2 + b\alpha + c = 0$ ...(i)

and $a^2\beta^2 - b\beta - c = 0$ ...(ii)

Now, $f(\alpha) = a^2\alpha^2 + 2b\alpha + 2c$

$= (a^2\alpha^2 + b\alpha + c) + (b\alpha + c)$ $[\because a^2\alpha^2 + b\alpha + c = 0]$

$= b\alpha + c = -a^2\alpha^2$

and $f(\beta) = a^2\beta^2 + 2b\beta + 2c$

$= 3b\beta + 3c = 3(b\beta + c) = 3a^2\beta^2$ $\left[\begin{matrix}\text{from Eq. (ii),}\\ a^2\beta^2 = b\beta + c\end{matrix}\right]$

$\because$ $0 < \alpha < \beta$

$\therefore$ α and β are real.

Now, $f(\alpha) < 0,\ f(\beta) > 0$

Hence, $f(\gamma) = 0$, where $\alpha < \gamma < \beta$

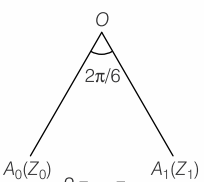
2. ***(c)*** Let $Z_0, Z_1, \cdots, Z_5$ be vertices respectively to centre O.

$|Z_0| = 1,\ A_0A_1 = |Z_1 - Z_0| = |Z_0e^{i\theta} - Z_0|$

$\therefore$ $A_0A_1 = |Z_0|\,|\cos\theta + i\sin\theta - 1|$

$= 1\sqrt{(\cos\theta - 1)^2 + (\sin^2\theta)} = \sqrt{2\,(1 - \cos\theta)}$

Now, $A_0A_1 = \sqrt{2 \cdot 2\sin^2\frac{\theta}{2}} = 2\sin\frac{\theta}{2}$



where, $\theta = \frac{2\pi}{6} = \frac{\pi}{3}$

On interchange θ by 2θ and 4θ, we get

$A_0A_1 = 2\sin\frac{2\theta}{2} = 2\sin\theta$ and $A_0A_4 = 2\sin\frac{4\theta}{2} = 2\sin 2\theta$

$\therefore$ $(A_0A_1)(A_0A_2)(A_0A_4) = 8\sin\frac{\pi}{6}\sin\frac{\pi}{3}\sin\frac{2\pi}{3}$

$= 8 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} = 3$

3. ***(c)*** The given sigma will contain $(n+1)$ terms out of which will be zero corresponding to $r = 0$

$\therefore$ $\sum = \left[\frac{1}{{}^nC_1} + \frac{2}{{}^nC_2} + \ldots + \frac{n-2}{{}^nC_{n-2}} + \frac{n-1}{{}^nC_{n-1}} + \frac{n}{{}^nC_n}\right]$

Now, apply ${}^nC_r = {}^nC_{n-r}$ and ${}^nC_n = 1 = {}^nC_0$

$1 + (n-1) = 2 + (n-2) = \ldots = n$

The last term is n and combine

$n\left[\frac{1}{{}^nC_1} + \frac{1}{{}^nC_2} + \ldots + \frac{1}{{}^nC_n}\right]$

$= n\left[\frac{1}{{}^nC_0} + \frac{1}{{}^nC_1} + \frac{1}{{}^nC_2} + \ldots\right]$ $[\because {}^nC_n = {}^nC_0]$

$= \frac{n}{2}\left[\frac{2}{{}^nC_0} + \frac{2}{{}^nC_1} + \ldots\right]$

$= \frac{n}{2}\left[\left(\frac{1}{{}^nC_0} + \frac{1}{{}^nC_n}\right) + \left(\frac{1}{{}^nC_1} + \frac{1}{{}^nC_{n-1}}\right) + \ldots\right]$

$= \frac{n}{2}\sum_{r=0}^{n}\frac{1}{{}^nC_r} = \frac{1}{2}na_n$

4. ***(c)*** $f(n) = \begin{cases}\frac{n-1}{2}, & \text{when } n \text{ is odd}\\ \frac{-n}{2}, & \text{when } n \text{ is even}\end{cases}$

and $f: N \to I$, where N is the set of natural numbers and I is the set of integers.

Let $x, y \in N$ and both are even.

Then, $f(x) = f(y)$

$\Rightarrow$ $\frac{-x}{2} = \frac{-y}{2} \Rightarrow x = y$

Again, let $x, y \in N$ and both are odd.

Then, $f(x) = f(y) \Rightarrow \frac{x-1}{2} = \frac{y-1}{2} \Rightarrow x = y$

So, mapping is one-one.

Since, each negative integer is an image of even natural number and positive integer is an image of odd natural number. So, mapping is onto. Hence, mapping is one-one and onto.

5. **(a)** Let $S = E(\theta)\,E(\phi)$

$$\Rightarrow S = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}\begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

$$\Rightarrow S = \begin{bmatrix} \cos\theta\cos\phi\cos(\theta-\phi) & \cos\theta\sin\phi\cos(\theta-\phi) \\ \cos\phi\sin\theta\cos(\theta-\phi) & \sin\theta\sin\phi\cos(\theta-\phi) \end{bmatrix}$$

$$\Rightarrow S = \begin{bmatrix} \cos\theta\cos\phi\cos(2n+1)\frac{\pi}{2} & \cos\theta\sin\phi\cos(2n+1)\frac{\pi}{2} \\ \cos\phi\sin\theta\cos(2n+1)\frac{\pi}{2} & \sin\theta\sin\phi\cos(2n+1)\frac{\pi}{2} \end{bmatrix}$$

$$\therefore \quad S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad [\because \theta - \phi = (2n+1)\frac{\pi}{2}, \text{ given}]$$

6. **(c)** The system of linear equations

$$a_1x + b_1y + c_1z = 0, \quad a_2x + b_2y + c_2z = 0$$

and $$a_3x + b_3y + c_3z = 0$$

has a non-zero solution, if $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = 0$

If system of linear equations has a non-zero solution, then

$$\begin{bmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{bmatrix} = 0$$

On applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get

$$\begin{bmatrix} 1 & 2a & a \\ 0 & 3b-2a & b-a \\ 0 & 4c-2a & c-a \end{bmatrix} = 0$$

$$\Rightarrow (3b-2a)(c-a) - (4c-2a)(b-a) = 0$$

$$\Rightarrow 3bc - 3ba - 2ac + 2a^2 = 4bc - 2ab - 4ac + 2a^2$$

$$\Rightarrow 4ac - 2ac = 4bc - 2ab - 3bc + 3ab$$

$$\Rightarrow 2ac = bc + ab$$

On dividing by abc, we get $\frac{2}{b} = \frac{1}{a} + \frac{1}{c}$

Hence, a, b and c are in HP.

7. **(b)** The number of ways in which atmost n books can be selected out of a collection of $(2n+1)$ books is

$${}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n$$

$$\Rightarrow \frac{1}{2}\{({}^{2n+1}C_1 + {}^{2n+1}C_1) + ({}^{2n+1}C_2 + {}^{2n+1}C_2) + \ldots + ({}^{2n+1}C_n + {}^{2n+1}C_n)\} = 63$$

$$\Rightarrow \{({}^{2n+1}C_1 + {}^{2n+1}C_{2n}) + ({}^{2n+1}C_2 + {}^{2n+1}C_{2n-1}) + \ldots + ({}^{2n+1}C_n + {}^{2n+1}C_{n+1})\} = 126$$

$$\Rightarrow {}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_{2n} = 126$$

$$\Rightarrow 2^{2n+1} - 2 = 126$$

$$\Rightarrow 2^{2n+1} = 128 = 2^7$$

$$\Rightarrow 2n+1 = 7$$

$$\Rightarrow n = 3$$

8. **(b)** $$\lim_{h\to 0^+}\frac{A}{P^3} = \lim_{h\to 0^+}\frac{h\sqrt{2hr-h^2}}{8[\sqrt{2hr-h^2}+\sqrt{2hr}]^3}$$

$$= \lim_{h\to 0}\frac{h\cdot\sqrt{h}\sqrt{2r-h}}{8\cdot\sqrt{h}\cdot h[\sqrt{2r-h}+\sqrt{2r}]^3}$$

$$= \lim_{h\to 0}\frac{\sqrt{2r-h}}{8[\sqrt{2r-h}+\sqrt{2r}]^3} = \frac{\sqrt{2r}}{8(2\sqrt{2r})^3} = \frac{1}{128r}$$

9. **(a)** The range of $\sin^{-1}x$ is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$.

Given, $\sin^{-1}x = 2\sin^{-1}a$

$$\frac{-\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}$$

$$\Rightarrow \frac{-\pi}{2} \le 2\sin^{-1}a \le \frac{\pi}{2}$$

$$\Rightarrow -\frac{\pi}{4} \le \sin^{-1}a \le \frac{\pi}{4}$$

$$\Rightarrow \sin\left(-\frac{\pi}{4}\right) \le a \le \sin\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \frac{-1}{\sqrt{2}} \le a \le \frac{1}{\sqrt{2}}$$

$$\Rightarrow |a| \le \frac{1}{\sqrt{2}}$$

10. **(a)** Let $P(\alpha, \beta)$ be the point which is equidistant to $A(a_1, b_1)$ and $B(a_2, b_2)$.

$$\therefore \quad (PA)^2 = (PB)^2$$

$$\Rightarrow (\alpha - a_1)^2 + (\beta - b_1)^2 = (\alpha - a_2)^2 + (\beta - b_2)^2$$

$$\Rightarrow \alpha^2 + a_1^2 - 2a_1\alpha + \beta^2 + b_1^2 - 2\beta b_1 = \alpha^2 + a_2^2 - 2\alpha a_2 + \beta^2 + b_2^2 - 2\beta b_2$$

$$\Rightarrow 2(a_2 - a_1)\alpha + 2(b_2 - b_1)\beta + (a_1^2 + b_1^2 - a_2^2 - b_2^2) = 0$$

$$\Rightarrow (a_2 - a_1)\alpha + (b_2 - b_1)\beta + \frac{1}{2}(a_1^2 + b_1^2 - a_2^2 - b_2^2) = 0$$

Thus, the equation of locus (α, β) is

$$(a_2 - a_1)x + (b_2 - b_1)y + \frac{1}{2}(a_1^2 + b_1^2 - a_2^2 - b_2^2) = 0$$

But the given equation is $(a_2 - a_1)x + (b_2 - b_1)y + c = 0$

$$\therefore \quad c = -\frac{1}{2}(a_1^2 + b_1^2 - a_2^2 - b_2^2) = \frac{1}{2}(a_2^2 + b_2^2 - a_1^2 - b_1^2)$$

11. **(c)** The intersection point of diameters of a circle is the centre of a circle.

The given equations of diameters are

$$3x - 4y - 7 = 0 \quad \ldots(i)$$

and $$2x - 3y - 5 = 0 \quad \ldots(ii)$$

On solving Eqs. (i) and (ii), we get

$$x = 1 \text{ and } y = -1$$

So, centre of circle is $(1, -1)$.

Let r be the radius of circle, then

$$\pi r^2 = 49\pi \Rightarrow r = 7 \text{ units}$$

Hence, the equation of required circle is

$$(x-1)^2 + (y+1)^2 = 49$$

$$\Rightarrow x^2 + y^2 - 2x + 2y + 1 + 1 = 49$$

$$\Rightarrow x^2 + y^2 - 2x + 2y - 47 = 0$$

12. **(a)** Since, $f(x)$ is continuous for all x, therefore it is continuous at $x = 1$ also.

$$\therefore \quad f(1) = \lim_{h\to 0} f(1-h)$$

$$\Rightarrow 1 = \lim_{h\to 0}[a(1-h)^2 + b]$$

$$\Rightarrow a + b = 1 \quad \ldots(i)$$

Also, $f(x)$ is differentiable at $x = 1$

$$\Rightarrow \lim_{h\to 0}\frac{f(1-h) - f(1)}{-h} = \lim_{h\to 0}\frac{f(1+h) - f(1)}{h}$$

$$\Rightarrow \quad \lim_{h\to 0}\frac{a(1-h)^2+b-1}{-h}=\lim_{h\to 0}\frac{\frac{1}{|1+h|}-1}{h}$$

$$\Rightarrow \quad \lim_{h\to 0}\frac{(a+b-1)+(h^2-2h)a}{-h}=\lim_{h\to 0}\frac{1-1-h}{h(1+h)}$$

$$\Rightarrow \quad 2a=-1 \qquad [\text{using } a+b=1]$$

$$\therefore \quad a=\frac{-1}{2}$$

From Eq. (i), $a+b=1\Rightarrow b=1-a=1+\frac{1}{2}=\frac{3}{2}$

Hence, $a=-\frac{1}{2}$ and $b=\frac{3}{2}$

13. ***(b)*** **Statement II** $(p\to q)\leftrightarrow(\sim q\to\sim p)=(p\to q)\leftrightarrow(p\to q)$ which is always true, so statement II is true.

Statement I $(p\wedge\sim q)\wedge(\sim p\wedge q)\equiv p\wedge\sim q\wedge\sim p\wedge q$

$\equiv p\wedge\sim p\wedge\sim q\wedge q\equiv f\wedge f\equiv f$

Hence, it is a fallacy statement. So, Statement I is true.

Alternative Solution

Statement II $(p\to q)\leftrightarrow(\sim q\to\sim p)$

$\sim q\to\sim p$ is contrapositive of $p\to q$

Hence, $(p\to q)\leftrightarrow(p\to q)$ will be a tautology.

Statement I $(p\wedge\sim q)(\sim p\wedge q)$

p	q	$\sim p$	$\sim q$	$p\wedge\sim q$	$\sim p\wedge q$	$(p\wedge\sim q)\wedge(\sim p\wedge q)$
T	T	F	F	F	F	F
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	F	T	T	F	F	F

Hence, it is a fallacy.

14. ***(b)*** We have, $f(x)=x^3+(a+2)x^2+3ax+5$

$$\Rightarrow \quad f'(x)=3x^2+2(a+2)x+3a$$

$f(x)$ is monotonic in R, if

$$f'(x)<0,\forall\, x\in R$$

or $$f'(x)>0,\forall\, x\in R$$

Since, the coefficient of $x^2=3>0$, therefore $f'(x)\not<0,\forall\, x\in R$

$$\therefore \quad f'(x)>0,\forall\, x\in R$$

$\therefore$ Discriminant $=4(a+2)^2-4\cdot 3\cdot 3a<0$

$$\Rightarrow \quad a^2-5a+4<0$$

$$\Rightarrow \quad (a-1)(a-4)<0$$

i.e. $$1<a<4$$

15. ***(d)*** Let $b=ar$ and $c=ar^2$, so that a, b and c are in GP.

$$\therefore \quad a+b+c=xb$$

$$\Rightarrow \quad a+ar+ar^2=x\cdot ar$$

$$\Rightarrow \quad r^2+(1-x)r+1=0 \qquad \ldots(i)$$

If r is real, then discriminant of Eq. (i) ≥ 0

$$\Rightarrow \quad (1-x)^2-4\cdot 1\cdot 1\geq 0$$

$$\Rightarrow \quad x^2-2x-3\geq 0$$

$$\Rightarrow \quad (x+1)(x-3)\geq 0$$

$$\Rightarrow \quad x\leq -1 \text{ or } x\geq 3$$

16. ***(c)*** Let $I=\sqrt{2}\int\frac{\sin x}{\sin\left(x-\frac{\pi}{4}\right)}dx$

Put $x-\frac{\pi}{4}=t\Rightarrow dx=dt$

$$\therefore \quad I=\sqrt{2}\int\frac{\sin\left(\frac{\pi}{4}+t\right)}{\sin t}dt=\sqrt{2}\int\left[\frac{1}{\sqrt{2}}\cot t+\frac{1}{\sqrt{2}}\right]dt$$

$$=t+\log|\sin t|+C=x+\log\left|\sin\left(x-\frac{\pi}{4}\right)\right|+C$$

17. ***(b)*** $I=\int_0^1\frac{\sin x}{\sqrt{x}}dx<\int_0^1\frac{x}{\sqrt{x}}dx$

because in $x\in(0,1)$, $x>\sin x$

$$I<\int_0^1\sqrt{x}\,dx=\frac{2}{3}[x^{3/2}]_0^1\Rightarrow I<\frac{2}{3}$$

and $$J=\int_0^1\frac{\cos x}{\sqrt{x}}dx<\int_0^1 x^{-1/2}dx=2\Rightarrow J<2$$

Hence, I $\frac{2}{3}$ and $J<2$

18. ***(c)*** $n(S)={}^{100}C_3$

Since, the AM of three numbers is 25.

$\therefore$ Their sum $=75$

$\therefore$ $n(E)=$ The number of integral solution of

$$x_1+x_2+x_3=75$$

where, $x_1\geq 1$, $x_2\geq 1$, $x_3\geq 1$

$$=\text{Coefficient of } x^{75} \text{ in } (x+x^2+x^3+\ldots)^3$$

$$=\text{Coefficient of } x^{72} \text{ in } (1+x+x^2+\ldots)^3$$

$$=\text{Coefficient of } x^{72} \text{ in}\left(\frac{1}{1-x}\right)^3$$

$$=\text{Coefficient of } x^{72} \text{ in } (1-x)^{-3}={}^{74}C_{72}={}^{74}C_2$$

$$\therefore \quad P(E)=\frac{{}^{74}C_2}{{}^{100}C_3}$$

19. ***(a)*** (λ,μ) is the point interior to both the curves, if $\lambda^2+\mu^2-16<0$ and $\mu^2-4\lambda<0$

Now, $$\mu^2-4\lambda<0\ \Rightarrow\ \lambda>\left(\frac{\mu}{2}\right)^2$$

If $\mu=0$, then $\lambda=1,2,3,\ldots$

If $\mu=1$, then $\lambda=1,2,3,\ldots$

If $\mu=2$, then $\lambda=2,3,\ldots$

If $\mu=3$, then $\lambda=3,4,\ldots$

Also, $\lambda^2+\mu^2-16<0$

$$\Rightarrow \quad \lambda^2<16-\mu^2$$

If $\mu=0$, then $\lambda=1,2,3$

If $\mu=1$, then $\lambda=1,2,3$

If $\mu=2$, then $\lambda=1,2,3$

If $\mu=3$, then $\lambda=2,3$

Now, λ has no integral value.

Hence, (1, 0), (2, 0), (3, 0),(1, 1), (2, 1), (3, 1), (2, 2), (3, 2) are the 8 possible points.

20. ***(d)*** $\because$ $(1,2)\in S$ but $(2,1)\notin S$

$\therefore$ S is not symmetric.

Hence, S is not equivalence relation.

Given, $$T=\{(x,y):x-y\in I\}$$

Now, $x-x=0\in I$, it is reflexive relation.

$$x-y\in I$$

$\Rightarrow y-x\in I$, it is symmetric relation.

Let $x-y=l_1$ and $y-z=l_2$

Now $x-z=(x-y)+(y-z)$

$=l_1+l_2\in l$

$\therefore$ T is also transitive.

Hence, T is an equivalence relation.

21. ***(d)*** Given, $\mathbf{b}=\hat{\mathbf{i}}+\hat{\mathbf{j}}$ and $\mathbf{c}=\hat{\mathbf{j}}+\hat{\mathbf{k}}$

The equation of bisector of $\mathbf{b}$ and $\mathbf{c}$ is

$$\mathbf{r}=\lambda(\mathbf{b}+\mathbf{c})=\lambda\left(\frac{\hat{\mathbf{i}}+\hat{\mathbf{j}}}{\sqrt{2}}+\frac{\hat{\mathbf{j}}+\hat{\mathbf{k}}}{\sqrt{2}}\right)=\frac{\lambda}{\sqrt{2}}(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}})\ ...(i)$$

Since, point D lies on bisector of $\mathbf{b}$ and $\mathbf{c}$.

$\therefore$ $$\frac{\lambda}{\sqrt{2}}(\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}})=(\hat{\mathbf{i}}+\hat{\mathbf{j}})+\mu(\hat{\mathbf{j}}+\hat{\mathbf{k}})$$

On equating the coefficient of $\hat{\mathbf{i}}$ both sides, we get

$$\frac{\lambda}{\sqrt{2}}=1\Rightarrow\lambda=\sqrt{2}$$

On putting $\lambda=\sqrt{2}$ in Eq. (i), we get

$$\mathbf{r}=\hat{\mathbf{i}}+2\hat{\mathbf{j}}+\hat{\mathbf{k}}$$

Since, the given vector $\mathbf{a}$ represents the same bisector equation $\mathbf{r}$

$\therefore$ $\alpha=1$ and $\beta=1$

22. ***(b)*** $|\cot x|=\begin{cases}-\cot x, & \text{if } \cot x<0\\ \cot x, & \text{if } \cot x\geq 0\end{cases}$

Case I $\cot x\geq 0$

$$\cot x=\cot x+\frac{1}{\sin x}\Rightarrow\frac{1}{\sin x}=0\Rightarrow\sin x=\infty$$

This is not possible.

Case II $\cot x<0$

$\Rightarrow$ $$-\cot x=\cot x+\frac{1}{\sin x}$$

$\Rightarrow$ $$2\cot x+\frac{1}{\sin x}=0$$

$\Rightarrow$ $2\cos x+1=0,\ \sin x\neq 0$

$\therefore$ $$\cos x=\frac{-1}{2}=-\cos\frac{\pi}{3}$$

$\Rightarrow$ $$\cos x=\cos\left(\pi+\frac{\pi}{3}\right)\text{ or }\cos\left(\pi-\frac{\pi}{3}\right)$$

$\therefore$ $$x=\frac{2\pi}{3},\frac{4\pi}{3}$$

23. ***(c)*** $\bar{x}=\dfrac{\text{Sum of quantities}}{n}=\dfrac{\frac{n}{2}(a+l)}{n}$

$$=\frac{101+d\left(\frac{100\times 101}{2}\right)}{101}=\frac{101(1+50d)}{101}$$

$$=\frac{50\times 101d+101}{101}=1+50d$$

$\therefore$ $$\text{MD}=\frac{1}{n}\Sigma|x_i-\bar{x}|=255$$

$$\Rightarrow\frac{1}{101}[50d+49d+48d+\ldots+d+0+d+\ldots+50d]=255$$

$\Rightarrow$ $$255=\frac{2d}{101}\left[\frac{50\times 51}{2}\right]$$

$\Rightarrow$ $$d=\frac{255\times 101}{50\times 51}$$

$\therefore$ $$d=10.1$$

24. ***(a)*** Here, $n(S)=1$ and length of the interval $[0,5]=5$

$n(E)=$ Length of the interval $\leq[0,5]$, in which P belongs such that the given equation has real roots.

Now, $x^2+Px+\frac{1}{4}(P+2)=0$ will have real roots, if

$$P^2-4\cdot 1\cdot\frac{1}{4}(P+2)\geq 0\Rightarrow P^2-P-2\geq 0$$

$\Rightarrow$ $$(P+1)(P-2)\geq 0\Rightarrow P\leq -1\text{ or }P\geq 2$$

But $P\in[0,5]$.

So, $$E=[2,5]$$

$\therefore n(E)=$ Length of the interval $[2,5]=3$

Hence, required probability $=\dfrac{3}{5}$

25. ***(a)*** Let AP be the electric pole of height h.

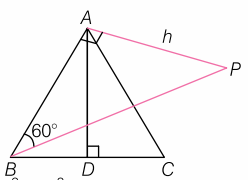
Then, $AB=h\cot 60^\circ=\dfrac{h}{\sqrt{3}}$

$\therefore$ $$AB=BC=AC=\frac{h}{\sqrt{3}}$$

In ΔABD, using Pythagoras theorem,

$$BD^2=AB^2-AD^2$$

$\Rightarrow$ $$\left(\frac{1}{2}\cdot\frac{h}{\sqrt{3}}\right)^2=\left(\frac{h}{\sqrt{3}}\right)^2-(100)^2$$



$\Rightarrow$ $$\frac{h^2}{3}-\frac{h^2}{12}=(100)^2$$

$\Rightarrow$ $$\frac{h^2}{4}=(100)^2$$

$\therefore$ $$h=200\text{ m}$$

26. ***(a)*** Using $c^2=a^2+b^2-2ab\cos C$, we have

$$c^2+2ab\cos C=101c^2$$

$\Rightarrow$ $$100c^2=2ab\cos C$$

$\Rightarrow$ $$ab\cos C=50c^2 \quad ...(i)$$

Also, $\dfrac{\cot C}{\cot A+\cot B}=\dfrac{\cos C\cdot\sin A\cdot\sin B}{\sin C\sin(A+B)}=\cos C\dfrac{\sin A\cdot\sin B}{\sin^2 C}$

$$=\cos C\cdot\frac{ab}{c^2} \quad [\text{using } \sin A=ak] \ ...(ii)$$

From Eq. (i), we get

$$\frac{\cot C}{\cot A+\cot B}=50$$

27. ***(d)*** $\because$ $[3u\ pv\ pw]-[pv\ w\ qu]+[2w\ qv\ qu]=0$

$\therefore 3p^2[\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})]-pq[\mathbf{v}\cdot(\mathbf{w}\times\mathbf{u})]+2q^2[\mathbf{w}\cdot(\mathbf{v}\times\mathbf{u})]=0$

$\Rightarrow$ $$(3p^2-pq-2q^2)[\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})]=0$$

But $[\mathbf{u}\ \mathbf{v}\ \mathbf{w}]\neq 0$

$\Rightarrow$ $$3p^2-pq-2q^2=0$$

$\Rightarrow$ $$p-q=0\text{ and }(3p+2q)=0$$

$\Rightarrow$ $$p=q\text{ and }p=\frac{-2q}{3}$$

28. **(c)** $R_1 = \int_{-1}^{2} x f(x)dx$

Using $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ ...(i)

$R_1 = \int_{-1}^{2}(1-x) f(1-x)dx$ [given $f(x) = f(1-x)$]

$\therefore$ $R_1 = \int_{-1}^{2}(1-x) f(x)dx$ ...(ii)

Given, R_2 is area bounded by $f(x)$, $x = -1$ and $x = 2$

$\therefore$ $R_2 = \int_{-1}^{2} f(x)dx$ ...(iii)

On adding Eqs. (i) and (ii), we get

$2R_1 = \int_{-1}^{2} f(x)dx$ ...(iv)

From Eqs. (iii) and (iv), we get

$2R_1 = R_2$

29. **(c)** Given, $\frac{x^2}{16} + \frac{y^2}{4} = 1$

Here, $a = 4$ and $b = 2$

Equation of normal

$4x\sec\theta - 2y\operatorname{cosec}\theta = 12$

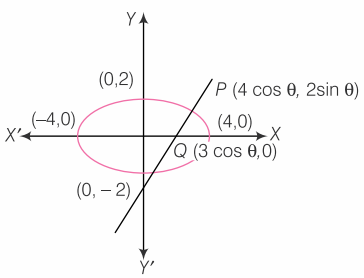
$M\left(\frac{7\cos\theta}{2}, \sin\theta\right) = (h, k)$

$\therefore$ $h = \frac{7\cos\theta}{2} \Rightarrow \cos\theta = \frac{2h}{7}$

and $k = \sin\theta$

$\Rightarrow$ $\frac{4h^2}{49} + k^2 = 1$ [$\because \cos^2\theta + \sin^2\theta = 1$]

Hence, locus is $\frac{4x^2}{49} + y^2 = 1$ ...(i)



For given ellipse, $e^2 = 1 - \frac{4}{16} = \frac{3}{4}$

$\therefore$ $e = \frac{\sqrt{3}}{2}$

$\therefore$ $x = \pm 4 \times \frac{\sqrt{3}}{2} = \pm 2\sqrt{3}$ [$\because x = \pm ae$] ...(ii)

On solving Eqs. (i) and (ii), we get

$\frac{4}{49} \times 12 + y^2 = 1$

$\Rightarrow$ $y^2 = 1 - \frac{48}{49} = \frac{1}{49}$

$\Rightarrow$ $y = \pm\frac{1}{7}$

$\therefore$ Required points $= \left(\pm 2\sqrt{3}, \pm\frac{1}{7}\right)$

30. **(a)** Given, $\frac{dy}{dt} - \frac{t}{(1+t)}y = \frac{1}{(1+t)}$

and $y(0) = -1$

which represent linear differential equation of first order

$\therefore$ IF $= e^{\int -\left(\frac{t}{1+t}\right)dt}$

$= e^{-t + \log(1+t)} = e^{-t} \cdot (1+t)$

Hence, required solution is

$ye^{-t}(1+t) = \int \frac{1}{1+t} e^{-t}(1+t)dt + C$

$= \int e^{-t}dt + C$

$\Rightarrow$ $ye^{-t}(1+t) = -e^{-t} + C$

$\because$ $y(0) = -1$

$\therefore$ $-1 \cdot e^0(1+0) = -e^0 + C$

$C = 0$

Now, $y = -\frac{1}{(1+t)}$

$\Rightarrow$ $y(1) = \frac{-1}{2}$

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TEST RIDER

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JEE MAIN RIDE 2

Instructions *This question paper contains 30 questions each having four options, out of which only one option is correct echoose the correct option.*

1. The function $f:(-\infty,-1]\to(0,e^5]$ defined by $f(x)=e^{x^3-3x+2}$ is

(a) many-one and onto (b) many-one and into
(c) one-one and onto (d) one-one and into

2. The solutions of the equation $z(\overline{z-2i})=2(2+i)$ are

(a) $3+i, 3-i$ (b) $1+3i, 1-3i$
(c) $1+3i, 1-i$ (d) $1-3i, 1+i$

3. The number of real roots of the equation $2^{\sin^2 x}-2^{\cos^2 x}=1$ is

(a) 2 (b) 1 (c) infinite (d) None of these

4. If $\begin{bmatrix}\alpha & \beta\\ \gamma & -\alpha\end{bmatrix}$ is to be the square root of two rowed unit matrix, then α, β and γ should satisfy the relation

(a) $1+\alpha^2+\beta\gamma=0$ (b) $1-\alpha^2-\beta\gamma=0$
(c) $1-\alpha^2+\beta\gamma=0$ (d) $\alpha^2+\beta\gamma-1=0$

5. The domain of the function

$f(x)=\sqrt[3]{1-3x}+3\cos^{-1}\left(\frac{2x-1}{3}\right)+e^{3\tan x}$ is

(a) $(-1, 2)$ (b) $[-1, 2]$ (c) $(-\infty, \infty)$ (d) None of these

6. Let α_1, α_2 and β_1, β_2 be the roots of $ax^2+bx+c=0$ and $px^2+qx+r=0$, respectively. If the system of equation $\alpha_1 y+\alpha_2 z=0$ and $\beta_1 y+\beta_2 z=0$ has a non-trivial solution, then

(a) $\frac{b^2}{q^2}=\frac{ac}{pr}$ (b) $\frac{c^2}{r^2}=\frac{ab}{pq}$
(c) $\frac{a^2}{p^2}=\frac{bc}{qr}$ (d) None of these

7. The number of positive integral solutions of $15<x_1+x_2+x_3\le 20$, is

(a) 785 (b) 685 (c) 1150 (d) 1250

8. If $\{x\}$ denotes the fractional part of x, then the value of $\left\{\frac{2^{2003}}{17}\right\}$ is

(a) $\frac{3}{17}$ (b) $\frac{5}{17}$ (c) $\frac{8}{17}$ (d) None of these

9. If $a+b+c=3$ and $a>0,\ b>0,\ c>0$, then the greatest value of $a^2b^3c^2$ is

(a) $\frac{3^{10}\cdot 2^4}{7^7}$ (b) $\frac{3^9\cdot 2^4}{7^7}$ (c) $\frac{3^8\cdot 2^4}{7^7}$ (d) $\frac{3^9\cdot 2^7}{7^7}$

10. If $f(x)=x-[x]$, where $[\cdot]$ denotes the greatest integer $\le x$ and $g(x)=\lim_{n\to\infty}\frac{\{f(x)\}^{2n}-1}{\{f(x)\}^{2n}+1}$, then $g(x)$ is equal to

(a) 0 (b) 1 (c) -1 (d) 2

11. If $f(x)=\left[\frac{1}{\sqrt{2}}(\cos x+\sin x)\right],\ 0<x<2\pi$, where $[\cdot]$ denotes the greatest integer function, then the number of points of discontinuity of $f(x)$ is

(a) 5 (b) 4 (c) 3 (d) 2

12. The solution set of $f'(x)g'(x)$, where $f(x)=\left(\frac{1}{2}\right)5^{2x+1}$ and $g(x)=5^x+4x\log 5$, is

(a) $(1, \infty)$ (b) $(0, 1)$ (c) $(0, \infty)$ (d) $[0, \infty)$

13. A function f is such that $f'(4)=f''(4)=0$ and f has minimum value 10 at $x=4$. Then, $f(x)$ is equal to

(a) $4+(x-4)^4$ (b) $10+(x-4)^4$
(c) $(x-4)^4$ (d) $(x-4)^4+12$

14. If $\phi(x)=\lim_{n\to\infty}\frac{x^n-x^{-n}}{x^n+x^{-n}},\ 0<x<1,\ n\in N$, then $\int \sin^{-1}x\,\phi(x)dx$ is equal to

(a) $x\sin^{-1}x+\sqrt{1-x^2}+C$ (b) $-(x\sin^{-1}x+\sqrt{1-x^2})+C$
(c) $x\sin^{-1}x-\sqrt{1-x^2}+C$ (d) None of these

15. In a statistical investigation of 1003 families of Kolkata, it was found that 63 families had neither a radio nor a TV, 794 families had a radio and 187 had a TV. The number of families in that group having both a radio and a TV is

(a) 36 (b) 32 (c) 41 (d) None of these

16. The planes $3x - y + z + 1 = 0$ and $5x + y + 3z = 0$ intersect in the line PQ. The equation of the plane through the point $(2, 1, 4)$ and perpendicular to PQ is

(a) $x + y - 2z = 5$ (b) $x + y - 2z = -5$
(c) $x + y + 2z = 5$ (d) $x + y + 2z = -5$

17. The base of a cliff is circular. From the extremities of a diameter of the base angles of elevation of the top of the cliff are 30° and 60°. If the height of the cliff is 500 m, then the diameter of the base of the cliff is

(a) $\frac{2000}{\sqrt{3}}$ m (b) $\frac{1000}{\sqrt{3}}$ m (c) $\frac{2000}{\sqrt{2}}$ m (d) $1000\sqrt{3}$ m

18. If $k \in N$ and $I_k = \int_{-2k\pi}^{2k\pi} |\sin x|[\sin x]dx$ (where $[\cdot]$ denotes the greatest integer function), then $\sum_{k=1}^{100} I_k$ is equal to

(a) –10100 (b) –40400
(c) 20200 (d) None of these

19. Area bounded by curves $y = [\cos A + \cos B + \cos C]$, $y = \left[7 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}\right]$ (where, $[\cdot]$ denotes the greatest integer function and A, B, C are angles of a triangle) and curve $|x - 4| + |y| = 2$ is

(a) 2 (b) 3 (c) 4 (d) 6

20. If the differential equation $\frac{d^2y}{dx^2} = \frac{g}{f} + c$, $c \in (0, \infty)$ and $f \neq 0$ represents a family of circles, then

(a) g, f have same sign (b) g, f have opposite sign
(c) $|g| < |f|$ (d) $|g| = |f|$

21. The point $P(2, 1)$ is shifted through a distance $3\sqrt{2}$ units measured parallel to the line $x + y = 1$ in the direction of decreasing ordinates, to reach at Q. The image of Q with respect to given line is

(a) (3, – 4) (b) (–3, 2) (c) (0, – 1) (d) None of these

22. The exhaustive range of values of a such that the angle between the pair of tangents drawn from (a, a) to the circle $x^2 + y^2 - 2x - 2y - 6 = 0$ lies in the range $\left(\frac{\pi}{3}, \pi\right)$, is

(a) (1, ∞) (b) (–5, – 3) ∪ (3, 5)
(c) $(-\infty, -2\sqrt{2}) \cup (2\sqrt{2}, \infty)$ (d) (–3, – 1) ∪ (3, 5)

23. If tangents PQ and PR are drawn from a point on the circle $x^2 + y^2 = 25$ to the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1, (b < 4)$, so that the fourth vertex S of parallelogram $PQSR$ lies on the circumcircle of ΔPQR, then eccentricity of the ellipse is

(a) $\frac{\sqrt{5}}{4}$ (b) $\frac{\sqrt{7}}{3}$ (c) $\frac{\sqrt{7}}{4}$ (d) $\frac{\sqrt{5}}{3}$

24. The line $\frac{x}{k} = \frac{y}{2} = \frac{z}{-12}$ makes an isosceles triangle with the planes $2x + y + 3z - 1 = 0$ and $x + 2y - 3z - 1 = 0$, then the value of k is

(a) 1 (b) –2 (c) 2 (d) 4

25. Let $\mathbf{a}$ and $\mathbf{c}$ be unit vectors and $|\mathbf{b}| = 4$ with $\mathbf{a} \times \mathbf{b} = 2\mathbf{a} \times \mathbf{c}$. The angle between $\mathbf{a}$ and $\mathbf{c}$ is $\cos^{-1}\left(\frac{1}{4}\right)$. If $\mathbf{b} - 2\mathbf{c} = \lambda\,\mathbf{a}$, then λ is equal to

(a) $\frac{1}{3}$ (b) $\frac{1}{4}$ (c) 4 (d) 3

26. If $\phi = \sin^{-1}(\sin\sqrt{z})$, where $z \in (80, 140)$ and z is cubes of first n natural numbers, then

(a) $\phi = (n + 1)\pi - 10$ (b) $\phi = (n - 1)\pi - 10$
(c) $\phi = (n + 1)\pi + 10$ (d) $\phi = (n - 1)\pi + 10$

27. If p, q and r are simple propositions with truth values T, F, T, then the truth value of $(\sim p \vee q) \wedge \sim r \Rightarrow p$ is

(a) true (b) false
(c) true, if r is false (d) true, if q is true

28. If in an obtuse angled triangle, the obtuse angle is $\frac{3\pi}{4}$ and the other two angles are equal to two values of θ satisfying $a\tan\theta + b\sec\theta = c$, where $|b| \leq \sqrt{a^2 + c^2}$, then $a^2 - c^2$ is equal to

(a) ac (b) $2ac$ (c) $\frac{a}{c}$ (d) None of these

29. If $f(x) = \begin{cases} x^3 - x^2 + 10x - 5, & x \leq 1 \\ -2x + \log_2(b^2 - 2), & x > 1 \end{cases}$, then the set of all possible values of b such that $f(x)$ has greatest value at $x = 1$, is

(a) $[-\sqrt{130}, -\sqrt{2}) \cup (\sqrt{2}, \sqrt{130}]$ (b) $[-\sqrt{140}, -\sqrt{3}) \cup (\sqrt{3}, \sqrt{140}]$
(c) $[-\sqrt{130}, \sqrt{130}]$ (d) None of these

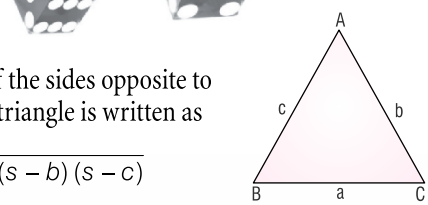
30. A tangent and a normal are drawn at the point $P \equiv (16, 16)$ of the parabola $y^2 = 16x$ which cut the axes of the parabola at the points A and B, respectively. If the centre of the circle through P, A and B is C, then the angle between PC and X-axis is

(a) $\tan^{-1}\left(\frac{1}{2}\right)$ (b) $\tan^{-1}(2)$ (c) $\tan^{-1}\left(\frac{3}{4}\right)$ (d) $\tan^{-1}\left(\frac{4}{3}\right)$

The solution of these questions are available on http://www.arihantbooks.com/Mathematics%20Spectrum.pdf

Revision through CONCEPT MAP

A Scientific way to learn quickly...



In a triangle ABC, the angles are denoted by capital letters A, B and C and the lengths of the sides opposite to these angles are denoted by small letters a, b and c respectively. Semi-perimeter of the triangle is written as $s = \frac{a+b+c}{2}$ and its area denoted by S of Δ. Area of triangle $\Delta = \sqrt{(s-a)(s-b)(s-c)}$

Some basic Rule

Sine rule

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} = \frac{1}{2R},$$

where R is the radius of the circum circle of the ΔABC.

Cosine rule

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac},$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Napier's analogy

$$\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\left(\frac{A}{2}\right),$$

$$\tan\left(\frac{C-A}{2}\right) = \frac{c-a}{c+a}\cot\left(\frac{B}{2}\right),$$

$$\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\left(\frac{C}{2}\right)$$

Auxiliary Formulae

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}},$$

$$\sin\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}},$$

$$\sin\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$$

$$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}},$$

$$\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}},$$

$$\cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$$

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}},$$

$$\tan\frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}},$$

$$\tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$$

Area of a Triangle

$$\Delta = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$$

$$= \sqrt{s(s-a)(s-b)(s-c)} = \frac{abc}{4R} = rs$$

where, R and r are the radii of the circum circle and the incircle of the ΔABC respectively.

Projection Rule

In any ΔABC,

$$a = b\cos C + c\cos B,\ b = c\cos A + a\cos C,$$

$$c = a\cos B + b\cos A$$

Solution of Triangle

The three sides , , and the three angles , and are called the elements of the . When any three of these six elements (except all the three angles) of a triangle are given, the triangle is known completely; that is the other three elements can be expressed in terms of the given elements and can be evaluated. This process is called the solution of triangles.

Three sides and one angle is given

If the three sides a, b, c are given, then $\angle A$ is obtained from

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \text{ or } \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Two sides *b* and *a* and the included angle is given

If two sides b and c and the included $\angle A$ are given, then

$$\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\left(\frac{A}{2}\right) \text{ gives } \frac{B-C}{2}.$$

Also $\frac{B+C}{2} = 90° - \frac{A}{2}$, so that B and C can be evaluated.

The third side is given by

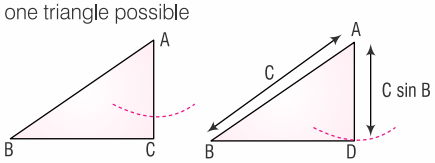
$$a = b\frac{\sin A}{\sin B} \text{ or } a^2 = b^2 + c^2 - 2bc\cos A$$

Two sides and opposite angle is given

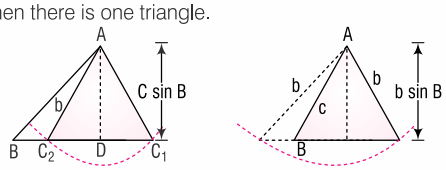
(i) If two sides b and c and the $\angle B$ (opposite to side b) are given, then $\sin C = \frac{c}{b}\sin B$, $A = 180° - (B + C)$ and $a = \frac{b\sin A}{\sin B}$ give the remaining elements

(ii) If $b < c\sin B$, then there is no triangle possible

If $b = c\sin B$ and B is an acute angle, then there is only one triangle possible



(iii) If $c\sin B < b < c$ and B is an acute angle, then there are two values of $\angle C$. If $c < b$ and B is an acute angle , then there is one triangle.

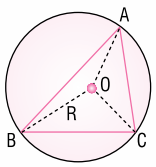


PROPERTIES AND SOLUTION OF TRIANGLE

Circle Connected with Triangle

Circumcircle

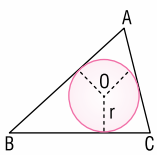
The circle passing through the vertices of the ΔABC is called the circumcircle. Its radius R is called the circumradius. In the ΔABC,



$$R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}$$

Incircle

Incircle The circle touching the three sides of the triangle internally is called the inscribed or the incircle of the triangle. Its radius r is called the inradius of the circle.



In the ΔABC,

$$r = \frac{\Delta}{s} = (s-a)\tan\frac{A}{2} = (s-b)\tan\frac{B}{2}$$

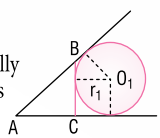
$$= (s-c)\tan\frac{C}{2} = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

From $r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$, we find that $r \le 4R \cdot \frac{1}{8}$

$\Rightarrow 2r \le R$. Here, equality holds for the equilateral triangle.

Escribed circle

The circle touching and the two sides AB and AC produced of triangle ABC externally is called the escribed circle opposite A. Its radius is denoted by r_1



Similarly, r_2 and r_3 denoted the radii of the escribed circles opposite angles B and C respectively. r_1, r_2, r_3 are called the exradii of ΔABC. Here,

$$r_1 = \frac{\Delta}{s-a} = s\tan\frac{A}{2} = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2},$$

$$r_2 = \frac{\Delta}{s-b} = s\tan\frac{B}{2} = 4R\sin\frac{B}{2}\cos\frac{C}{2}\cos\frac{A}{2},$$

$$r_3 = \frac{\Delta}{s-c} = s\tan\frac{C}{2} = 4R\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2},$$

$$r_1 + r_2 + r_3 = 4R + r,$$

$$r_1r_2 + r_2r_3 + r_3r_1 = s^2 = \frac{r_1r_2r_3}{r}$$

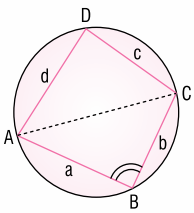
Polygon

Cyclic Quadrilateral

(i) Area of the cyclic quadrilateral

$= \sqrt{(s-a)(s-b)(s-c)(s-d)}$

when $2s = a + b + c + d$



(ii) $\cos B = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}$.

(iii) Radius of the circumcircle

$$= \frac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(ad+bc)}{(s-a)(s-b)(s-c)(s-d)}}$$

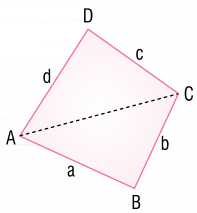
Quadrilateral

Area of the quadrilateral

$\Delta^2 = (s-a)(s-b)(s-c)(s-d) - abcd\cos\alpha$

Corollary I If $d = 0$, then the quadrilateral becomes a triangle.

Corollary II The quadrilateral, whose sides are given, has greatest area when it can be inscribed in a circle.



Regular Polygon

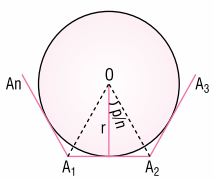
Let $A_1, A_2, \ldots, A_n$ be a regular polygon of n sides each of length a.

(i) **Inscribed circle of a regular polygon of *n* sides**

Area $= nr^2\tan\frac{\pi}{n}$

Radius, $r = \frac{a}{2}\cot\frac{\pi}{n}$

fig.

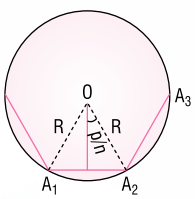


(ii) **Circumscribed circle of a regular polygon of *n* sides**

Area $= \frac{nR^2}{2}\sin\frac{2\pi}{n}$

Radius $= R = \frac{a}{2}\operatorname{cosec}\frac{\pi}{n}$

fig.



PEDAL TRIANGLE

- Triangle formed by jolming the feet of altitudes is called pedal triangle.
- Important fact of Pedal trangle
- Orthocentre of the triangle is the incentre of the pedal triangle.
- If , and be the centres of escribed circles which are opposite to A, B and C respectively and l the centre of incircle, then triangle ABC is the pedal triangle of the trangle , , and l is the orthocentre of the triangle , .
- The centroid of the triangle lies on the line joining the circumcentre to the orthocentre and divides it in the ratio 1:2.
- Circle circumscribing the pedal triangle of a given triangle bisects the sides of the given triangle and also the lines joining the vertices of the given triangle to orthocentre of the given triangle. This circle is known as nine point circle.
- Circumecentre of the pedal triangle of a given triangle bisects the line joining the circumcentre of the triangle to teh orthocentre.

TEST RIDER

The Simulator test series towards JEE Main and Advanced

JEE ADVANCED RIDE 1

Paper 1

Section 1 (One or More Than One Option Correct Type)

1. A function $f(x)$ is defined in the interval [1, 4] as follows $f(x) = \begin{cases} \log_e[x], & 1 \le x < 3 \\ |\log_e x|, & 3 \le x \le 4 \end{cases}$, then the graph of $f(x)$

(a) is broken at two points
(b) is broken at exactly one point
(c) does not have a definite tangent at two points
(d) does not have a definite tangent at more than two points

2. $f(x)$ is cubic polynomial which has local maximum at $x = -1$. If $f(2) = 18$, $f(1) = -1$ and $f'(x)$ has local minima at $x = 0$, then

(a) the distance between point of maxima and minima is $2\sqrt{5}$
(b) $f(x)$ is increasing for $x \in [1, 2\sqrt{5})$
(c) $f(x)$ has local minima at $x = 1$
(d) the value of $f(0) = 5$

3. System of equations

$$x + 3y + 2z = 6$$
$$x + \lambda y + 2z = 7$$
$$x + 3y + 2z = \mu \text{ has}$$

(a) unique solution, if $\lambda = 2, \mu \neq 6$
(b) infinitely many solution, if $\lambda = 4, \mu = 8$
(c) no solution, if $\lambda = 5, \mu = 7$
(d) no solution, if $\lambda = 3, \mu = 5$

4. If $\mathbf{a}$ and $\mathbf{b}$ are unit vectors with included angle $\frac{\pi}{6}$, then

(a) angle between $((\mathbf{a}\cdot\mathbf{b})\mathbf{a} - \mathbf{b})$ and $\mathbf{b}\times(\mathbf{a}\times\mathbf{b})$ is $\pi/6$
(b) angle between $((\mathbf{a}\cdot\mathbf{b})\mathbf{a} - \mathbf{b})$ and $\mathbf{b}$ is $2\pi/3$
(c) angle between $((\mathbf{a}\cdot\mathbf{b})\mathbf{a} - \mathbf{b})$ and $(\mathbf{a} - (\mathbf{a}\cdot\mathbf{b})\mathbf{b})$ is $\pi/3$
(d) angle between $\mathbf{a}$ and $(\mathbf{a} - (\mathbf{a}\cdot\mathbf{b})\mathbf{b})$ is $\pi/3$

5. If in a four-dimensional space unit vectors along axes are $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ and $\hat{\mathbf{l}}$ and $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ and $\mathbf{a}_4$ are four non-zero vectors such that no vector can be expressed as linear combination of others and $(\lambda - 1)(\mathbf{a}_1 - \mathbf{a}_2) + \mu(\mathbf{a}_2 + \mathbf{a}_3) + \lambda(\mathbf{a}_3 + \mathbf{a}_4 - 2\mathbf{a}_2) + \mathbf{a}_3 + \delta\mathbf{a}_4 = 0$, then

(a) $\lambda = 1$ *(b)* $\mu = -\frac{2}{3}$ *(c)* $\lambda = \frac{2}{3}$ *(d)* $\delta = \frac{1}{3}$

6. Let $p\lambda^4 + q\lambda^3 + r\lambda^2 + s\lambda + t = \begin{vmatrix} \lambda^2 + 3\lambda & \lambda - 1 & \lambda + 3 \\ \lambda + 1 & -2\lambda & \lambda - 4 \\ \lambda - 3 & \lambda + 4 & 3\lambda \end{vmatrix}$ be an identity in λ, where p, q, r, s and t are constants. Then, the value of t is greater than

(a) 2 *(b)* -2 *(c)* 4 *(d)* -4

7. An even polynomial function $f(x)$ satisfies a relation $f(2x)\left(1 - f\left(\frac{1}{2x}\right)\right) + f(16x^2y) = f(-2) - f(4xy), \forall\ x, y \in R - \{0\}$ and $f(4) = -255$, $f(0) = 1$.

Which of the following hold(s) good?

(a) $f(x)$ has local maximum at $x = 1$
(b) $f(x)f\left(\frac{1}{x}\right) \le 0$
(c) Range of values of k for which $|f(x)| = k - 2$ has exactly four distinct solutions is (2, 3)
(d) $\int_0^1 f(x)\,dx = \frac{3}{4}$

8. Let $f(x)$ be a polynomial function defined from R to R with leading coefficient 2 and degree 5 such that $f(1) = 2$, $f(2) = 3$, $f(3) = 4$, $f(4) = 5$ and $f(5) = 6$, then which of the following hold(s) good?

(a) $f(x)$ is many-one onto function
(b) $f(x)$ is one-one onto function
(c) $f(6) = 127$
(d) $f(0) = -239$

9. Let $S_1 = 0$ and $S_2 = 0$ are two circles and they are mirror image w.r.t. the line $L = 0$. If the centre of $S_1 = 0$ is (2, 4) and the point of intersection of their internal common tangents is (6, 8), then which of the following is true?

(a) Centre of $S_2 = 0$ is (10, 12)

(b) Sum of intercepts made by the line $L = 0$ on coordinate axes is 28

(c) Point of intersection of their external common tangent does not exist

(d) Radical axis of $S_1 = 0$ and $S_2 = 0$ is $L = 0$

10. If x and y are independent variables and $f(x) = 1 + \int_0^1 (x^2y^2 + x)\, f(y)\, dy$ be a continuous and differentiable function, then identify the correct option(s)

(a) $f'(0) = \frac{164}{57}$ (b) $f'(0) = \frac{25}{19}$

(c) $\int_0^1 f(x)\, dx = \frac{164}{57}$ (d) $\int_0^1 f(x)\, dx = \frac{25}{19}$

Section 2 (One Integer Value Correct Type)

11. Let f be a function defined on R (the set of all real numbers) such that $f'(x) = 2010(x - 2009)(x - 2010)^2 (x - 2011)^3 (x - 2012)^4$ for all $x \in R$. If g is a function defined on R with values in the interval $(0, \infty)$ such that $f(x) = \quad (g(x))$, for all $x \in R$, then the number of points in R at which g has local maximum is

12. Number of solutions of the equation $\cot^{-1}\sqrt{x^2 + 2x} + \cos^{-1}\sqrt{x^2 + 2x + 1} = \frac{\pi}{2}$ is

13. Iet $f(x) = \lim_{n \to \infty} \sum_{n=1}^{n} 3^n \sin^3\left(\frac{x}{3^n}\right)$ and $g(x) = x - 4f(x)$. If the value of $\lim_{x \to 0}(1 + g(x))\cot x$ is λ, then λ is equal to

14. A differentiable function f satisfies the relation $f(x + y) + f(xy - 1) = f(x) + f(y) + f(xy)$, $\forall x, y \in R$. If $f(1) = 2, f'(0) = 1$ and $f'(-1) = -1$ find $f(2)$.

15. Let ω be the complex number $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. Then, the number of distinct complex number z satisfying $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$ is equal to

16. If $\sum_{i=0}^{m} {}^{20}C_{20-i}\ {}^{30}C_{m-i}$, where $m < 30$, is maximum and minimum, when $m = a$ and $m = b$ respectively, then value of $\left[\frac{a+b}{6}\right]$ is (where [.] denotes the greatest integer function) (take ${}^nC_r = 0, \forall\ n < r$)

17. If $\int_{\cos x}^{1} t^2 f(t)dt = 1 - \cos x, \forall\ x \in \left(0, \frac{\pi}{2}\right)$, then the value of $\left[f\left(\frac{\sqrt{3}}{4}\right)\right]$ is equal to (where [.] denotes the greatest integer function)

18. Given that $\mathbf{u} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$; $\mathbf{v} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 4\hat{\mathbf{k}}$; $\mathbf{w} \quad \hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and $(\mathbf{u}\cdot\mathbf{R} - 15)\hat{\mathbf{i}} + (\mathbf{v}\cdot\mathbf{R} - 30)\hat{\mathbf{j}} + (\mathbf{w}\cdot\mathbf{R} - 25)\hat{\mathbf{k}} = 0$. Then, find the greatest integer less than or equal to $|\mathbf{R}|$.

19. The largest term of the sequence $\frac{1}{503}, \frac{4}{524}, \frac{9}{581}, \frac{16}{692}, \ldots$ is $\frac{49}{1529p}$. Then, the value of p is

20. A and B play a game of tennis. The situation of the game is as follows, if one scores two consecutive points after a deuce he wins, if loss of a point is followed by win of a point, it is deuce. The change of a server to win a point is 2/3. The game is at deuce and A is serving. Probability that A will win the match is p/q (p and q are coprime), then $p + q$ is (serves are changed after each game)

Paper 2

Section 1 (Only One Option Correct Type)

1. If the equation $k\sin^2 x + |\cos x| - 2k = 0$ has atleast one real solution, then the exhaustive set of values of k is/are

(a) $\left(0, \frac{1}{2}\right)$ (b) $\left[-\frac{1}{2}, 0\right)$ (c) $\left[0, \frac{1}{2}\right]$ (d) $\left(0, \frac{1}{2}\right]$

2. A pair of numbers is picked up randomly (without replacement) from the set {1, 2, 3, 5, 7, 11, 12, 13, 17, 19}. The probability that the number 11 was picked given that the sum of the numbers was even, is nearly

(a) 0.1 (b) 0.125 (c) 0.24 (d) 0.18

3. If P is a point on the altitude AL of ΔABC such that $\angle PBC = \frac{B}{3}$, then the relation between $AP(y)$ and $BP(x)$ is

(a) $x^2 + y^2 = 2c^2$ (b) $x^2 + cy = c^2$

(c) $x^2 + cy^2 = c^2$ (d) None of these

4. A new chess game is invented which has 9×9 grid, tiled by 81 squares of size 1×1. In chess board pattern, instead of 8×8 and rule is that in a single step, the king can move only along horizontal or vertical adjacent tile. If king is at the centre tile of this grid, then total number of shortest paths he can take to reach at corner of chess board, are

(a) 260 (b) 140 (c) 280 (d) None of these

5. If α and $\beta, (\alpha > \beta)$ are roots of the equation $x^2 - \sqrt{2}x + \sqrt{3-2\sqrt{2}} = 0$, then the value of $(\cos^{-1}\alpha + \tan^{-1}\alpha + \tan^{-1}\beta)$ is

(a) $\frac{3\pi}{8}$ (b) $\frac{5\pi}{8}$ (c) $\frac{7\pi}{8}$ (d) $\frac{\pi}{3}$

6. The value of the definite integral $\int_0^{\pi}[(1 - x\sin 2x)e^{\cos^2 x} + (1 + x\sin 2x)e^{\sin^2 x}]dx$ is

(a) π (b) πe (c) $\pi(e - 1)$ (d) $\pi(e + 1)$

7. Let $f(x) = \begin{cases} \dfrac{e^{x^2} - \frac{2}{\pi}\sin^{-1}\sqrt{1-x}}{\ln(1+\sqrt{x})}; & x \in (0,1). \\ k & ; x = 0 \end{cases}$

If $f(x)$ is continuous at $x = 0$, then the value of k is

(a) $1 + \frac{2}{\pi}$ (b) $1 - \frac{2}{\pi}$

(c) $\frac{2}{\pi}$ (d) $\frac{-2}{\pi}$

8. A circle is drawn circumscribing an equilateral triangle with side of length 2 with its centroid at origin. A square is drawn circumscribing the circle. Then, the equation of a circle circumscribing the square is

(a) $x^2 + y^2 = 8$ (b) $3x^2 + 3y^2 = 8$

(c) $5x^2 + 5y^2 = 8$ (d) $x^2 + y^2 = 12$

9. If the variable line $y = ax + b$ is tangent to an ellipse $3x^2 + 4y^2 = 6$, then locus of point $P(a, b)$ is a conic, whose eccentricity is equal to

(a) $\sqrt{3}$ (b) $\sqrt{\frac{3}{2}}$

(c) $\sqrt{1+\sqrt{2}}$ (d) $\sqrt{1+\frac{1}{\sqrt{2}}}$

10. A curve is given by $y = \begin{cases} \sqrt{4-x^2}, & 0 \le x < 1 \\ \sqrt{3x}, & 1 \le x \le 3 \end{cases}$.

The area lying between the curve and the X-axis is equal to

(a) $\frac{(2\pi - \sqrt{3} + 36)}{6}$ (b) $\frac{(2\pi - \sqrt{3} + 36)}{12}$

(c) $\frac{(2\pi - \sqrt{3} + 36)}{3}$ (d) None of these

Section 2 (Comprehension Type)

This section contains three paragraphs, each describing theory, experiments, data, etc. Six questions relate to the three paragraphs with two questions on each paragraph. Each question has only one correct answer among the four given options (a), (b), (c) and (d).

Paragraph for Questions 11 and 12

A is to play with B in a three set match. For a particular set, the probability of A winning the set is y and if he wins, the probability of his winning the next set becomes $\sqrt{y}$, else the probability that he wins the next one becomes y^2. There is no probability that a set is to be abandoned. R is the probability that A wins the first set.

11. If $R = \frac{1}{2}$, then the probability that match will end in first two sets, is nearly equal to

(a) 0.73 (b) 0.95 (c) 0.51 (d) 0.36

12. If A losses the first set, then the value of R, such that his probability of winning the match is still larger than that of his lossing, is given by

(a) $R \in \left(\frac{1}{2}, 1\right)$ (b) $R \in \left(\left(\frac{1}{2}\right)^{\frac{1}{3}}, 1\right)$

(c) $R \in \left(\left(\frac{1}{2}\right)^{\frac{3}{2}}, 1\right)$ (d) no value of R

Paragraph for Questions 13 and 14

Tangent line to the curve $\left(\frac{x+1}{6}\right)^2 - \frac{y^2}{2} = 1$, passing from the point $A(3, 2)$ cuts the circle $x^2 + y^2 - 4x - 6y + 9 = 0$ at points P and Q. Further tangents drawn at P and Q intersect at R.

13. Equation of tangent to the hyperbola from A may be

(a) $x + y = 1$ (b) $3y = 2x + 2 - \sqrt{6}$

(c) $3x - 5y + 1 = 0$ (d) $2y = x + 1$

14. If tangent to the hyperbola through point A makes negative y-intercept, then coordinates of R is

(a) (1, 4)

(b) (4, 1)

(c) (– 4, –1)

(d) (4, – 1)

Paragraph for Questions 15 and 16

Let f be a function defined by $f(k) = \int_0^{\pi/2} |k - 2\sin t|\cos t\, dt$, where k is any real number.

15. Which one of the following statement is correct?

(a) $f(k)$ is continuous, $\forall\, k$ but not derivable at two values of k

(b) $f(k)$ is differentiable, $\forall\, k$

(c) $f(k) = 1 - k$, $\forall\, k \ge 2$

(d) $f(k) = k - 1$, $\forall\, k \ge 0$

16. The minimum value of $f(k)$ is

(a) $\frac{1}{2}$ (b) $\frac{3}{2}$

(c) 0 (d) 1

Section 3 (Matching List Type)

This section contains four questions, each having two matching lists. Choices for the correct combination of elements from Column I and Column II are given as options (a), (b), (c) and (d) out of which one is correct.

17.

Column I	Column II
P. If A and B are two points on parabola $y^2 = 16(x + 2)$ such that tangent at A is parallel to normal at B to parabola, then chord AB always passes through fixed point whose coordinates are (a, b), where $(a + b)$, is	1. –2
Q. When $(3x + 5)^{100}$ is expanded, the largest power of 2 dividing the coefficient of x^{39} is	2. 6
R. If z_0 lies on the curve $z\bar{z} - (1 - i)z - (1 - i)\bar{z} - 2 = 0$, then $\min\|z_0 - 5 - i\|$ is equal to	3. 2
S. Square of the length of the intercept cut off by the line $1 - x = y - 1 = 10 - z$ between the planes $\mathbf{r}\cdot\mathbf{n} = 3$ and $\mathbf{r}\cdot\mathbf{n} = -3$, where $\mathbf{n} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$ is	4. 3

Codes

	P	Q	R	S		P	Q	R	S		P	Q	R	S		P	Q	R	S
(a)	1	2	3	4	(b)	2	3	1	4	(c)	4	3	2	1	(d)	2	4	1	3

18.

Column I	Column II
P. If one root of the equation $2x^2 - 16x + k = 0$ lies in (3, 4) and the other in (4, 5), then number of integral value of k is	1. 6
Q. If $f(x) = e^x - e^{-x} - 2\sin x - \frac{2}{3}x^3$, then the least value of n for which $\left(\frac{d^n f(x)}{dx^n}\right)_{\text{at } x = 0}$ is non-zero, is	2. 5
R. A point P lies in the plane $2x + 3y + 5z = 1$ is situated at the minimum distance from the line segment AB. Coordinates of A and B are $(1, 0, -3)$ and $(1, -5, 7)$. If the distance between A and P is $\lambda\sqrt{\mu}$, then the value of $\frac{(\lambda + 1)\mu}{3}$ is	3. 7
S. If the line $\lambda x + 2y + \mu = 0$ is the bisector of $\angle A$ in the triangle, where vertices are $A(-8, 5)$, $B(-15, -19)$ and $C(1, -7)$, then the value of $\frac{\mu - (\lambda + 1)}{\lambda}$ is	4. 1

Codes

	P	Q	R	S		P	Q	R	S		P	Q	R	S		P	Q	R	S
(a)	1	2	3	4	(b)	2	3	1	4	(c)	1	3	2	4	(d)	4	3	2	1

19.

Column I	Column II
P. Let $f(x)$ be a continuous function on R. If $\int_0^1 [f(x) - f(2x)]dx = 5$ and $\int_0^2 [f(x) - f(4x)]dx = 10$, then the value of $\int_0^1 [f(9x) - f(8x)]dx$ is	1. 2
Q. Let $\mathbf{a}$ and $\mathbf{b}$ be two non-collinear vectors such that $\|\mathbf{a}\| = 2, \|\mathbf{b}\| = 1\cdot 5$, then possible integral value of the area of triangle whose two sides are represented by the vectors $\mathbf{a}$ and $\mathbf{a} \times \mathbf{b}$ is	2. 5
R. The value of λ for which $\frac{\int_0^{\pi/2}(\sin x + \lambda\cos x)dx}{\sqrt{\int_0^{\pi/2}(\sin x + \lambda\cos x)^2 dx}}$ is maximum, lies in the interval (a, b), then the value of $[(a + b)]$, where $[\cdot]$ denotes the greatest integer function $\le x$, is	3. 10
S. A classroom test of Mathematics consists of 5 objective type questions, each having 4 possible answers, out of which only one is correct. For each right answer, a student get 1 mark and for each wrong option – 1 mark. If the probability of getting atleast 2 marks answering to every question selected randomly is p/q, where p and q are coprimes in their lowest form, then the value of $\frac{p + q}{17}$ is	4. 3

Codes

	P	Q	R	S		P	Q	R	S		P	Q	R	S		P	Q	R	S
(a)	3	2	1	4	(b)	3	4	1	2	(c)	1	3	2	4	(d)	4	3	2	1

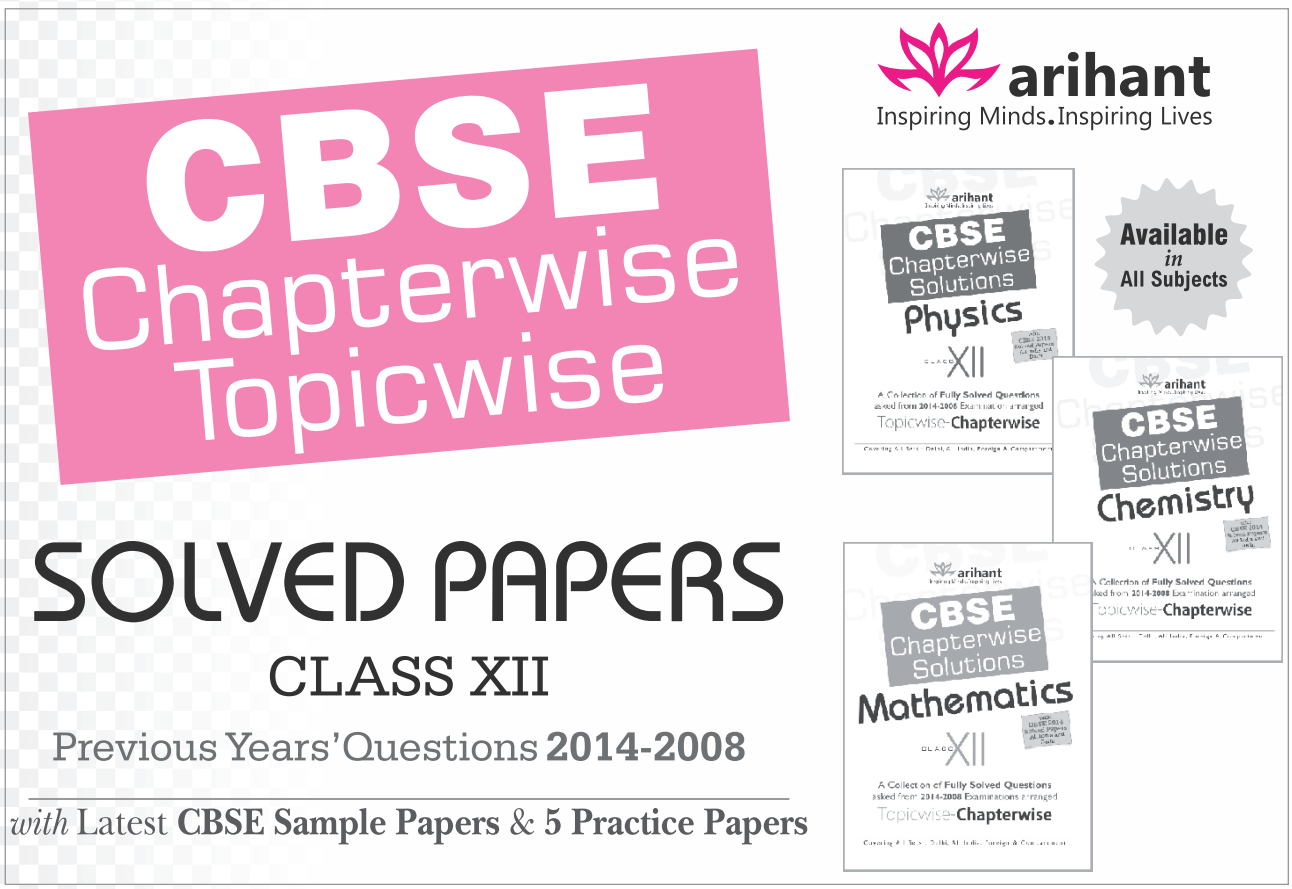
20.

	Column I	Column II
P.	A line lies in the plane $\mathbf{r}\cdot(\hat{\mathbf{i}}+3\hat{\mathbf{j}}-\hat{\mathbf{k}})=9$, is perpendicular to the line $\mathbf{r}=(\hat{\mathbf{i}}+\hat{\mathbf{j}}+\hat{\mathbf{k}})+\lambda(2\hat{\mathbf{i}}+\hat{\mathbf{j}}-\hat{\mathbf{k}})$ and passing through the point, where plane intersects the given line, has its equation $\mathbf{r}=(3\hat{\mathbf{i}}+2\hat{\mathbf{j}})+\mu(l\hat{\mathbf{i}}+m\hat{\mathbf{j}}+n\hat{\mathbf{k}})$, where $l+m+n$ is equal to	1. 2
Q.	$f(x)=\begin{cases}\left[\sin\left(\frac{2x^2}{a}\right)+\cos\left(\frac{3x}{b}\right)\right]^{\frac{ab}{x^2}}, & x\neq 0\\ e^{x^2-2x+3}, & x=0\end{cases}$ is continuous at $x=0$, $\forall\, b\in R$, then $\lvert 1/a_{\min}\rvert$ is equal to	2. 4
R.	The curve such that the intercept on the X-axis cut off between the origin and the tangent at a point is twice the abscissa and which passes through the point (1, 2). If the ordinate of the point on the curve is $\frac{1}{3}$, then the value of the abscissa is	3. 8
S.	If the sum of the series $\frac{5}{1^2\cdot 4^2}+\frac{11}{4^2\cdot 7^2}+\frac{17}{7^2\cdot 10^2}+\ldots$ is $\frac{p}{q}$, then $\frac{q-p}{p}$ is equal to	4. 6

Codes

	P	Q	R	S		P	Q	R	S		P	Q	R	S		P	Q	R	S
(a)	3	2	1	4	(b)	3	4	1	2	(c)	3	2	4	1	(d)	4	3	2	1

The solution of these questions are available on http://www.arihantbooks.com/Mathematics%20Spectrum.pdf



Naveen Chandra Joshi

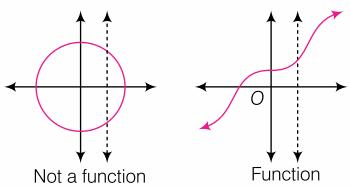
GRAPHS of FUNCTIONS AND THEIR APPLICATIONS

Coverage of Special topics for JEE Advanced

Graphical way to analyse the function helps us to get solution quickly without going in minute details of solution. Here, we are giving some basic techniques to draw the graphs and their use in different topics. Students are advised to understand these techniques properly to minimise time in exams.

Graphical Identification of Function

If we draw graph of the curve and a line is drawn parallel to *Y*-axis. If it intersect the curve at one point, then it is a function, otherwise it is not a function.

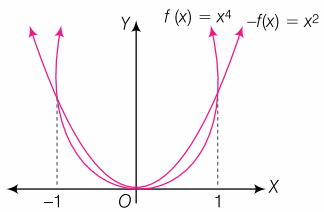


Some Basic Graphs

Student should well acquinted with the graphs such as identity function, quadratic function, modulus, greatest, least integer function, signum function and trigonometric function. We are introducing some of general graphs, then we will discuss transformation of the graphs, which will help us to know how we can change or transform the basic graph in different condition.

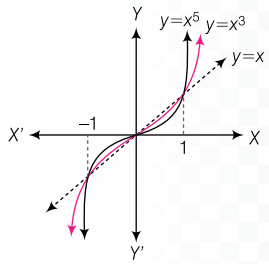
(i) **Graph of** $f(x) = x^{2n}$, $n \in N$

If $n \in N$, then function f given by $f(x) = x^{2n}$ is an even function, so its graph is always symmetric about *Y*-axis.



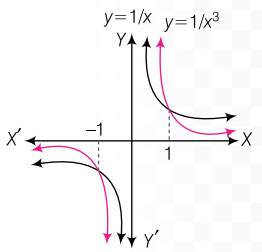
- $x^2 > x^4 > x^6 > \ldots\ldots, \forall\ x \in (-1, 1)$
- $x^2 < x^4 < x^6 < \ldots, \forall\ x \in (-\infty, -1) \cup (1, \infty)$

(ii) **Graph of** $f(x) = x^{2n-1}$ If $n \in N$, then the function f given by $f(x) = x^{2n-1}$ is an odd function. So, its graph is symmetric about origin or opposite quadrants.



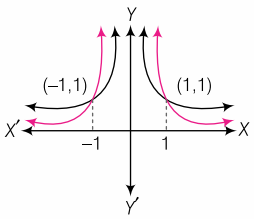
- $x > x^3 > x^5 > \ldots, \forall\ x \in (-1, 1)$
- $x < x^3 < x^5 < \ldots, \forall\ x \in (-\infty, -1) \cup (1, \infty)$

(iii) **Graph of** $f(x) = \dfrac{1}{x^{2n-1}}$ Since, $f(x)$ is an odd function, so its graph is symmetrical in opposite quadrant.



(iv) **Graph of** $f(x) = \frac{1}{x^{2n}}$; $n \in N$

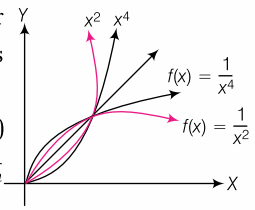
We observe that the function $f(x) = \frac{1}{x^{2n}}$ is an even function, so its graph is symmetric about Y-axis.



(v) **Irrational Function**

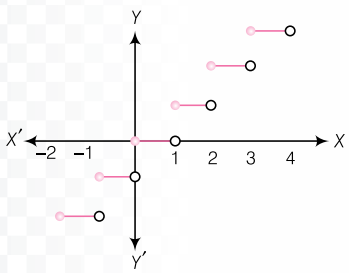
Here, $f(x) = x^{\frac{1}{2n}}$ is defined for all $x \in [0, \infty)$ and the values taken by $f(x)$ are positive.

So, domain and range of $f(x)$ are $[0, \infty)$. Graph of $f(x) = x^{\frac{1}{2n}}$ is mirror image of graph $f(x) = x^{2n}$ about the line $y = x$.



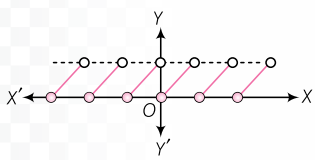
(vi) **Greatest Integer Function**

$f(x) = [x]$ indicates the integral part of x which is nearest and smaller integer to x. It is known as floor of x.



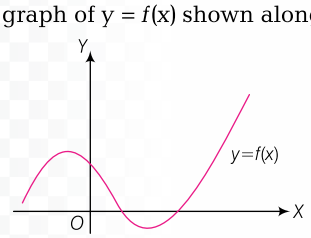
Fractional Part Function

Here, $\{\cdot\}$ denotes the fractional part of x. Thus, if $y = \{2 \cdot 5\} = 0 \cdot 5$. Also, $x = [x] + \{x\}$, where, $I = [x]$ and $f = \{x\}$.



Drawing the Graph of Transformation of a Function

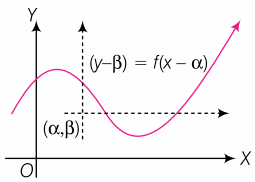
Consider, the graph of $y = f(x)$ shown alongside.



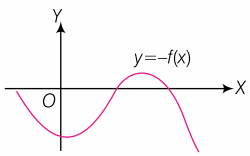
Using above basic graph, now we will learn how to find the graphs in different cases of $f(x)$. When there are some transformation which is introduced in $f(x)$, then read the cases given below :

(i) The graph of $(y - \beta) = f(x - \alpha)$

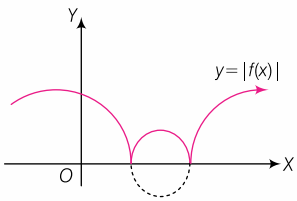
To get the graph of above function, shift the graph to (α, β) and then translate the graph of y with respect to the new axes.



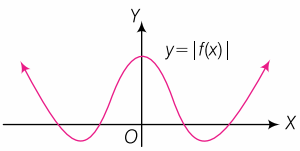
(ii) The graph $y = -f(x)$ drawn by taking the mirror image of the graph of $y = f(x)$ in X-axis or flip the graph along X-axis.



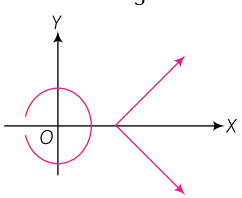
(iii) Graph $y = |f(x)|$ can be drawn by taking mirror image of the graph which lies below the X-axis and retaining original graph above X-axis.



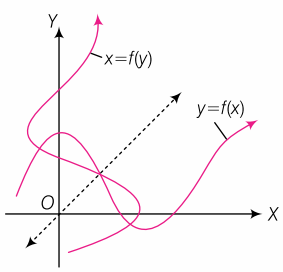
(iv) The graph of $y = f(|x|)$ is drawn by deleting the graph of $f(x)$ which lies left of Y-axis and taking the mirror image of the graph lying right of Y-axis.



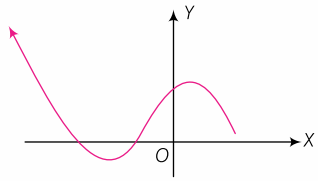
(v) The graph of $|y| = f(x)$ is drawn by deleting those portions of the graph which lies below X-axis and then taking mirror image of the remaining portion in the X-axis, as shown in figure.



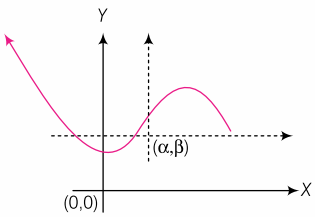
(vi) The graph of $x = f(y)$ is drawn by taking the mirror image of $y = f(x)$ in the line $y = x$ as shown in figure.



(vii) The graph of $y = f(-x)$ is drawn by taking the mirror image of the graph of $y = f(x)$ in the Y-axis as shown in figure

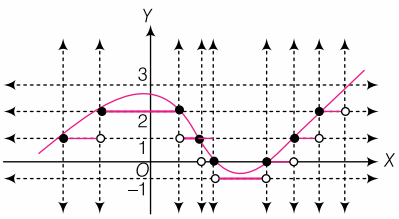


(viii) The graph $(y - \beta) = f\{-(x - \alpha)\}$ is drawn by taking mirror image of $f(x)$ with respect to Y-axis and then shifting the origin at (α, β) and then translating the graph $y = f(-x)$ with respect to new axes.

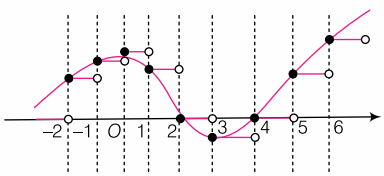


(ix) The graph of $y = [f(x)]$ (where, $[\cdot]$ denotes the greatest integer function)

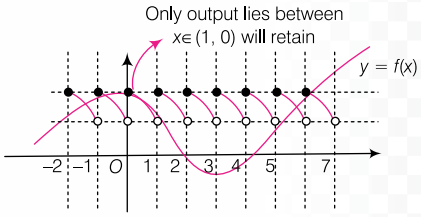
In order to draw $y = [f(x)]$ mark the integer on Y-axis. Draw the horizontal lines through the integrals till they intersect the graph. Draw vertical dotted lines from these intersection points finally draw horizontal line parallel to X-axis from any intersection point to the nearest vertical dotted lines with blank dot at right end.



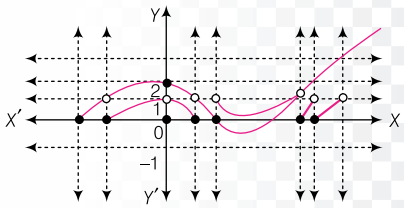
(x) From the graph $y = f(x)$ to draw graph $y = f([x])$. Since, input is an integer, to draw the graph mark the integers on the X-axis. Draw vertical lines till they intersect the graph of $f(x)$ from these intersection points draw horizontal lines to meet the nearest right vertical line.



(xi) From the graph $y = f(x)$ to draw graph $y = f(\{x\})$, to obtain graph of $y = f(\{x\})$. We retain graph of $f(x)$ for the value of x in between $[0, 1)$ and repeat it in the period of 1.



(xii) From the graph $y = f(x)$ to draw $y = \{f(x)\}$, here $\{\cdot\}$ represents fractional part function. Hence, output is always lies between $[0, 1)$. To plot the graph draw horizontal lines for all integral values of y, from intersection point draw vertical lines and shift (translate) the graph in between $y \in [0, 1)$.



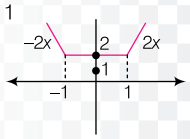
Application of Graphs

Here, we are giving examples in which we will use the graph to find the solution of questions based on continuity differentiability, domain and range. Graphical approach reduce time in great extent and it also helps to visualise the problem.

Example 1 If $f(x) = |x - 1| + |x + 1|$, then discuss the continuity and differentiability of the function.

Sol. $$f(x) = |x - 1| + |x + 1| = \begin{cases} (x - 1) + (x + 1) & , \quad x > 1 \\ -(x - 1) + (x + 1) & , \quad -1 \le x \le 1 \\ -(x - 1) - (x + 1) & , \quad x < -1 \end{cases}$$

$$f(x) = \begin{cases} 2x & , \quad x > 1 \\ 2 & , \quad -1 \le x \le 1 \\ -2x & , \quad x < -1 \end{cases}$$



From the graph, it is clear that function is continuous at all points. Also, at $x = \pm 1$ there is a sharp edges, hence function is not differentiable at $x = \pm 1$.

Here, at $x = 1$, RHD = 2, LHD = 0

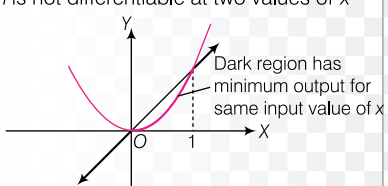
At $x = -1$, LHD = −2, RHD = 0

Example 2 Let $h(x) = \min\{x, x^2\}$ for every real number of x, then [IIT JEE 1998]

(a) h is not continuous for all x (b) h is not differentiable for all x

(c) $h'(x) = 1$ for all x (d) h is not differentiable at two values of x

Sol. ***(d)*** For $h(x) = \min\{x, x^2\}$, we plot the graph $y = x$ and $y = x^2$ and for $\min\{x, x^2\}$ consider that part of the graph which has smaller output for same input value of x.



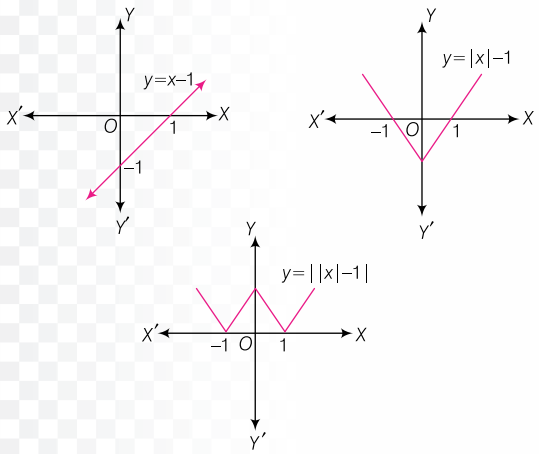
Which shows that h is continuous but not differentiable at two values of x (0 and 1).

Example 3 If $f(x) = ||x| - 1|$, then point where $f(x)$ is not differentiable, is/are [IIT JEE 2005]

(a) 0, ±1 (b) ±1
(c) 0 (d) 1

Sol. ***(a)*** We have, $f(x) = ||x| - 1|$

Draw graph $y = x - 1$, then transform it into $y = |x| - 1$ and find $f(x) = ||x| - 1|$, observe the graph formed, if sharp edges are there, then function is not differential at those points.



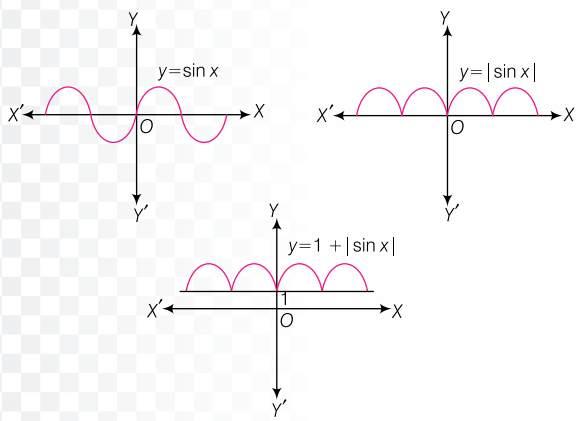
Hence, at $x = -1$, 0 and 1 function is not differentiable.

Example 4 The function $f(x) = 1 + |\sin x|$ is

(a) continuous nowhere
(b) continuous everywhere
(c) differentiable at $x = 0$
(d) not differentiable at infinite many points

Sol. ***(d)*** To check continuity and differentiability, we need to draw the graph using transformation.

1. Draw $y = \sin x$
2. Transform to $y = |\sin x|$
3. Then, translate 1 unit above the origin to get $f(x) = 1 + |\sin x|$



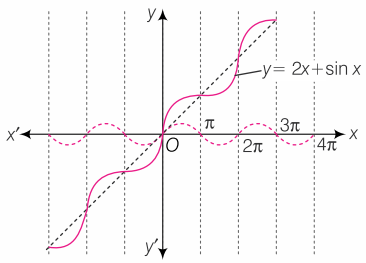
Example 5 Let $f : R \to R$ be defined by $f(x) = 2x + \sin x, \forall x \in R$, then f is [IIT JEE 2002]

(a) one-one and onto (b) one-one but not onto
(c) onto but not one-one (d) neither one-one nor onto

Sol. ***(a)*** To draw $f(x) = 2x + \sin x$,

first draw the graphs $y = 2x$ and $y = \sin x$

then to draw $f(x)$ move the graph of $y = 2x$ upward, when $\sin x > 0$ and downward, when $\sin x < 0$.



From the graph, it is clear that $f(x)$ is one-one and onto.

Note Function is one-one, if a line drawn parallel to X-axis intersect the graph only once and onto when codomain is equal to range of the function.

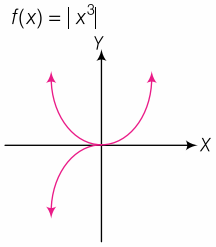
Example 6 Match the following :

	Column I		Column II
A.	$f(x) = \|x^3\|$ is	p.	continuous in $(-1, 1)$
B.	$f(x) = \sqrt{\|x\|}$ is	q.	differentiable in $(-1, 1)$
C.	$f(x) = \|\sin^{-1} x\|$ is	r.	differentiable in $(0, 1)$
D.	$f(x) = \cos^{-1}\|x\|$ is	s.	not differentiable atleast at one point in $(-1, 1)$

Sol. A → (p, q, r), B → (p, r, s), C → (p, r, s), D → (p, r, s)

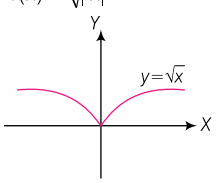
Let us draw graph of above function to get whether they are differentiable or not in said intervals.

A. $f(x) = |x^3|$



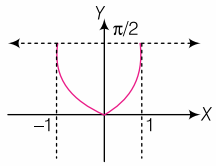
This is continuous and differentiable.

B. $f(x) = \sqrt{|x|}$



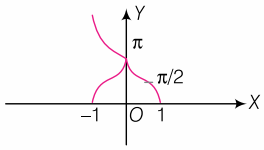
Clearly, from the graph $f(x)$ is non-differentiable at $x = 0$.

C. $f(x) = |\sin^{-1} x|$



$f(x)$ is not differentiable at $x = 0$.

D. $y = \cos^{-1}|x|$

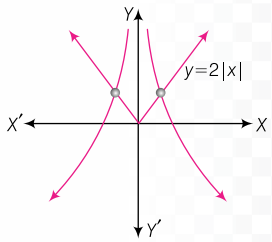


$f(x)$ is non-differentiable at $x = 0$.

Example 7 The number of real solutions of the equation $\log_{0.5}|x| = 2|x|$

(a) 1 (b) 2
(c) 0 (d) None of these

Sol. ***(b)*** Draw graph of $y = \log_{0.5}|x|$ and $y = 2|x|$



By graph, it is clear that it has only two solutions.
[$\because y = \log_{0.5}|x|$ and $y = 2|x|$ intersect at 2 points]

Smart Practice

Only One Correct Answer

1. If $f(x) = \sqrt{|x| - \{x\}}$ (where, $\{\cdot\}$ denotes the fractional part of x) and x, y are its domain and range respectively, then

(a) $x \in \left(-\infty, \frac{1}{2}\right]$ and $y \in \left[\frac{1}{2}, \infty\right)$

(b) $x \in \left(-\infty, -\frac{1}{2}\right] \cup [0, \infty)$ and $y \in \left[\frac{1}{2}, \infty\right)$

(c) $x \in \left(-\infty, -\frac{1}{2}\right] \cup [0, \infty)$ and $y \in [0, \infty)$

(d) None of the above

2. If $f(x) = x + 2|x+1| + 2|x-1|$. If $f(x) = k$ has exactly one real solution, then the value of k is

(a) 3 (b) 0
(c) 1 (d) 2

3. The number of the solutions of $2\cos x = |\sin x|, 0 \le x \le 4\pi$ is

(a) 0 (b) 2
(c) 4 (d) infinite

4. Let $f : R \to R$ and $g : R \to R$ be two one-one and onto functions such that they are the mirror image of each other about the line $y = a$. If $h(x) = f(x) + g(x)$, then $h(x)$ is

(a) one-one and onto (b) only one-one and not onto
(c) only onto but not one-one (d) neither one-one nor onto

5. Let $f_1(x) = \begin{cases} x, 0 \le x \le 1 \\ 1, x > 1 \\ 0, R - [0,1] \end{cases}$, $f_2(x) = f_1(-x), \forall x$,

$f_3(x) = -f_2(x), \forall x$

$f_4(x) = f_3(-x), \forall x$

Which of the following is necessarily true?

(a) $f_4(x) = f_1(x), \forall x$ (b) $f_1(x) = -f_3(-x), \forall x$
(c) $f_2(-x) = f_4(x), \forall x$ (d) $f_1(x) + f_3(x) = 0, \forall x$

6. The sum of roots of the equation $\cos^{-1}(\cos x) = [x]$, $[\cdot]$ denotes the greatest integer function, is

(a) $2\pi + 3$ (b) $\pi + 3$
(c) $\pi - 3$ (d) $2\pi - 3$

7. Discuss the differentiability of $f(x) = \max\{x^2 - 3x + 2, 2 - |x-1|\}$, then number of points of discontinuity is

(a) 1 (b) 2 (c) 3 (d) infinite

8. Discuss the differentiability of $f(x) = \max\{2\sin x, 1 - \cos\}, \forall x \in (0, \pi)$, then number of points of discontinuity

(a) 1 (b) 2
(c) 3 (d) 0

More than One Correct Answer

9. $f(x) = [|x|]$, where $[\cdot]$ denotes the greatest integer function, then which of the following is not true?

(a) $f(x)$ is continuous, $\forall\ x \in R$
(b) $f(x)$ is continuous from right and discontinuous from left, $\forall\ x \in I$
(c) $f(x)$ is continuous from left and discontinuous from right, $\forall\ x \in I$
(d) $f(x)$ is continuous at $x = 0$

10. $f(x) = \dfrac{[x]+1}{\{x\}+1}$ for $f : \left[0, \dfrac{5}{2}\right) \to \left(\dfrac{1}{2}, 3\right]$,

where $[\cdot]$ represents the greatest integer function and $\{\cdot\}$ represents the fractional part of x, then which of the following is true?

(a) $f(x)$ is injective discontinuous function
(b) $f(x)$ is surjective non-differentiable function
(c) $\min\left(\lim\limits_{x \to 1^-} f(x), \lim\limits_{x \to 1^+} f(x)\right) = f(1)$
(d) max (x values of point of discontinuity) = $f(1)$

11. The function $f(x) = \begin{cases} 1, |x| \geq 1 \\ \dfrac{1}{n^2}, \dfrac{1}{n} < |x| < \dfrac{1}{n-1}, n = 2, 3, \ldots \\ 0, x = 0 \end{cases}$

(a) $f(x)$ is discontinuous in infinite many points
(b) $f(x)$ is continuous everywhere
(c) $f(x)$ is discontinuous only at $x = \dfrac{1}{n}$, $n \in Z - \{0\}$
(d) None of the above

Match the Column

12. Match the following :

	Column I		Column II
A.	$x^2 \tan x = 1, x \in [0, 2\pi]$	p.	5
B.	$2^{\cos x} = \lvert\sin x\rvert, x \in [0, 2\pi]$	q.	2
C.	If $f(x)$ is a polynomial of degree 5 with real coefficient such that $f(\lvert x\rvert) = 0$ has 8 real roots, then the number of roots of $f(x) = 0$	r.	3
D.	$7^{\lvert x\rvert}(\lvert 5 - \lvert x\rvert\rvert) = 1$	s.	4

Integer Type Questions

13. Number of points of non-differentiability of function $f(x) = \max\{\sin^{-1}|\sin x|, \cos^{-1}|\sin x|\}, 0 < x < 2\pi$ is

14. Let $f : R \to R$ be a continuous onto function satisfying $f(x) + f(-x) = 0, \forall\ x \in R$.

If $f(-3) = 2$ and $f(5) = 4$ in $[-5, 5]$, then minimum number of roots of the equation $f(x) = 0$ is

15. Suppose that f is an even periodic function with period 2 and that $f(x) = x$, for all x in the interval $[0, 1]$, the value of $[10f(3.14)]$ is (where $[.]$ represents the greatest integer function.)

Answers

1. (c) **2.** (a) **3.** (c) **4.** (d) **5.** (b) **6.** (a) **7.** (c)
8. (a) **9.** (a, c) **10.** (a, b, d) **11.** (a, c) **12.** A $\longrightarrow$ q; B $\longrightarrow$ s; C $\longrightarrow$ p; D $\longrightarrow$ s
13. (7) **14.** (3) **15.** (8)

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VECTOR AND 3-D GEOMETRY

Coverage of Best Questions for JEE Advanced

Single Option Correct Type

1. If $\mathbf{p}$ and $\mathbf{q}$ are non-zero vectors such that none of them can be expressed as a scalar multiple of the other and satisfy $(2\cos\theta + 1)\,\mathbf{p} + (\sqrt{3}\cot\theta + 1)\,\mathbf{q} = 0$, then the most general values of θ are

(a) $n\pi + \frac{2\pi}{3}$ *(b)* $2n\pi + \frac{2\pi}{3}$

(c) $n\pi + \frac{5\pi}{6}$ *(d)* $2n\pi \pm \frac{5\pi}{6}$

Sol. ***(b)*** $\cos\theta = -\frac{1}{2}$ and $\cot\theta = -\frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{2\pi}{3}$

$\therefore$ Most general values $= 2n\pi + \frac{2\pi}{3}, n \in I$

2. Let $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ are three unit vectors in a plane such that they are equally inclined to each other and if $k_1 = (\mathbf{v}_1 \times \mathbf{v}_2)\cdot(\mathbf{v}_2 \times \mathbf{v}_3)$, $k_2 = (\mathbf{v}_2 \times \mathbf{v}_3)\cdot(\mathbf{v}_3 \times \mathbf{v}_1)$ and $k_3 = (\mathbf{v}_3 \times \mathbf{v}_1)\cdot(\mathbf{v}_1 \times \mathbf{v}_2)$, then the value of $(k_1 + k_2 - k_3)$ is equal to

(a) $\frac{-3}{4}$ *(b)* $\frac{3}{4}$ *(c)* $\frac{6}{4}$ *(d)* $\frac{9}{4}$

Sol. ***(b)*** Since, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are in a plane and equally inclined to each other, then angle between any two vectors is 120° and $\mathbf{v}_1 \times \mathbf{v}_2$, $\mathbf{v}_2 \times \mathbf{v}_3$ and $\mathbf{v}_3 \times \mathbf{v}_1$ are parallel.

$$(\mathbf{v}_1 \times \mathbf{v}_2)\cdot(\mathbf{v}_2 \times \mathbf{v}_3) = |\mathbf{v}_1 \times \mathbf{v}_2|\,|\mathbf{v}_2 \times \mathbf{v}_3|\cos 0°$$

$$= \frac{\sqrt{3}}{2} \times \frac{\sqrt{3}}{2} = \frac{3}{4}$$

Similarly, $(\mathbf{v}_2 \times \mathbf{v}_3)\cdot(\mathbf{v}_3 \times \mathbf{v}_1) = (\mathbf{v}_3 \times \mathbf{v}_1)\cdot(\mathbf{v}_1 \times \mathbf{v}_2) = \frac{3}{4}$

$\therefore$ $k_1 + k_2 - k_3 = \frac{3}{4}$

3. If $\mathbf{a}$ and $\mathbf{b}$ are two non-zero vectors, then the value of the scalar $[(\mathbf{a} \times \mathbf{b}) \times \mathbf{a}]\cdot[(\mathbf{b} \times \mathbf{a}) \times \mathbf{b}]$ equals

(a) $-(\mathbf{a}\cdot\mathbf{b})\,|\mathbf{a} \times \mathbf{b}|^2$ *(b)* $\mathbf{a}^2\,|\mathbf{a} \times \mathbf{b}|^2$

(c) $|\mathbf{b}|^2\,|\mathbf{a} \times \mathbf{b}|^2$ *(d)* $(\mathbf{a}\cdot\mathbf{b})\,|\mathbf{a} \times \mathbf{b}|^2$

Sol. ***(a)*** $[(\mathbf{a} \times \mathbf{b}) \times \mathbf{a}]\cdot[(\mathbf{b} \times \mathbf{a}) \times \mathbf{b}]$

$= [(\mathbf{a}\cdot\mathbf{a})\mathbf{b} - (\mathbf{b}\cdot\mathbf{a})\mathbf{a}]\cdot[(\mathbf{b}\cdot\mathbf{b})\mathbf{a} - (\mathbf{a}\cdot\mathbf{b})\mathbf{b}]$

$= (\mathbf{a}^2\mathbf{b}^2)\mathbf{a}\cdot\mathbf{b} - \mathbf{a}^2\mathbf{b}^2(\mathbf{a}\cdot\mathbf{b}) - (\mathbf{a}\cdot\mathbf{b})\mathbf{a}^2\mathbf{b}^2 + (\mathbf{a}\cdot\mathbf{b})^3$

$= -(\mathbf{a}\cdot\mathbf{b})\,|\mathbf{a} \times \mathbf{b}|^2$

4. If $\mathbf{u} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{v} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\mathbf{w} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$, then the value of $(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) - (\mathbf{w} \times \mathbf{v})\cdot[\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\,\mathbf{u}]$ is

(a) 0 *(b)* −1 *(c)* 1 *(d)* 2

Sol. ***(a)*** $(\mathbf{v}\cdot\mathbf{u})\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) - (\mathbf{w}\times\mathbf{v})\cdot[\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{u}]$

$= (\mathbf{v}\cdot\mathbf{u})[\mathbf{u}\ \mathbf{v}\ \mathbf{w}] + (\mathbf{u}\cdot\mathbf{v})[\mathbf{w}\ \mathbf{v}\ \mathbf{u}] = 0$, which vanishes.

5. If $\mathbf{c} = |\mathbf{a} \times \mathbf{b}|\,\mathbf{a} + (\mathbf{a}\cdot\mathbf{b})\,\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are unit vectors and angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\pi}{6}$, then the value of $|\mathbf{c}|$ is

(a) $\frac{\sqrt{5}}{2}$ *(b)* $\frac{\sqrt{3}}{2}$ *(c)* $\frac{\sqrt{1}}{2}$ *(d)* $\frac{\sqrt{7}}{2}$

Sol. ***(d)*** $\mathbf{c} = |\mathbf{a} \times \mathbf{b}|\,\mathbf{a} + (\mathbf{a}\cdot\mathbf{b})\,\mathbf{b}$

$|\mathbf{a}| = 1 = |\mathbf{b}|$

Angle between $\mathbf{a}$ and $\mathbf{b} = \frac{\pi}{6}$

$\therefore$ $|\mathbf{a} \times \mathbf{b}| = (1)(1)\sin\frac{\pi}{6} = \frac{1}{2}$

and $\mathbf{a}\cdot\mathbf{b} = (1)(1)\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$

$\Rightarrow$ $\mathbf{c} = \frac{1}{2}\mathbf{a} + \frac{\sqrt{3}}{2}\mathbf{b}$

$\Rightarrow \quad |2\,\mathbf{c}|^2 = |\mathbf{a} + \sqrt{3}\mathbf{b}|^2$

$\Rightarrow \quad 4|\mathbf{c}|^2 = 1 + 3 + 2\sqrt{3}\left(\frac{\sqrt{3}}{2}\right) = 7$

$\therefore \quad 4|\mathbf{c}|^2 = 7 \Rightarrow |\mathbf{c}| = \frac{\sqrt{7}}{2}$

More than One Option Correct Type

6. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-coplanar vectors. If the vectors $\mathbf{V}_1 = \mathbf{a} + 2\mathbf{b} + 3\mathbf{c}$, $\mathbf{V}_2 = \lambda\mathbf{b} + 4\mathbf{c}$ and $\mathbf{V}_3 = (2\lambda - 1)\,\mathbf{c}$ are non-coplanar (where, λ is a scalar), then λ cannot be

(a) 0 *(b)* $\frac{1}{2}$ *(c)* 1 *(d)* $\frac{3}{2}$

Sol. ***(c, d)*** $[\mathbf{V}_1\ \mathbf{V}_2\mathbf{V}_3] = \begin{vmatrix} 1 & 2 & 3 \\ 0 & \lambda & 4 \\ 0 & 0 & 2\lambda - 1 \end{vmatrix} \cdot [\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

$= 1[\lambda\,(2\lambda - 1)]\,[\mathbf{a}\ \mathbf{b}\mathbf{c}] \neq 0$

$= \lambda\,(2\lambda - 1)\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}] \neq 0$

As $[\mathbf{a}\,\mathbf{b}\mathbf{c}] \neq 0$, hence $[\mathbf{V}_1\ \mathbf{V}_2\ \mathbf{V}_3] \neq 0$

$\Rightarrow \quad \lambda\,(2\lambda - 1) \neq 0$

Hence, $\mathbf{V}_1$, $\mathbf{V}_2$ and $\mathbf{V}_3$ are non-coplanar, $\forall\ \lambda \in R$ except $\lambda = 0$ or $\lambda = \frac{1}{2}$.

7. Vectors $\mathbf{a} = 2\hat{\mathbf{i}} + (\alpha + \beta + \gamma + \delta)\,\hat{\mathbf{j}} + (\alpha\beta + \gamma\,\delta)\,\hat{\mathbf{k}}$

$\mathbf{b} = (\alpha + \beta + \gamma + \delta)\hat{\mathbf{i}} + 2\,(\alpha + \beta)\cdot(\gamma + \delta)\hat{\mathbf{j}} + [\alpha\,\beta\,(\gamma + \delta)$

$+ \gamma\delta\,(\alpha + \beta)]\,\hat{\mathbf{k}}$ and $\mathbf{c} = (\gamma\delta + \alpha\beta)\hat{\mathbf{i}} + [\alpha\,\beta\,(\gamma + \delta) +$

$\gamma\delta\,(\alpha + \beta)]\hat{\mathbf{j}} + 2\alpha\,\beta\gamma\delta$ are coplanars, then $\alpha, \beta, \gamma, \delta$ can be in

(a) AP *(b)* GP *(c)* equal *(d)* unequal

Sol. ***(a, b, c, d)***

$$\begin{vmatrix} 2 & \alpha+\beta+\gamma+\delta & \alpha\beta+\gamma\delta \\ \alpha+\beta+\gamma+\delta & 2(\alpha+\beta)(\gamma+\delta) & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) \\ \gamma\,\delta+\alpha\beta & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) & 2\alpha\beta\gamma\delta \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ (\alpha+\beta) & (\gamma+\delta) & 0 \\ \alpha\beta & \gamma\delta & 0 \end{vmatrix} \begin{vmatrix} 1 & (\gamma+\delta) & \gamma\delta \\ 1 & (\alpha+\beta) & \alpha\beta \\ 0 & 0 & 0 \end{vmatrix} = 0$$

For every values of $\alpha, \beta, \gamma, \delta$.

8. Length of the intercept cut off by the line $x - 1 = 1 - y = z - 10$ between the planes $\mathbf{r}\cdot\mathbf{a} = 3$ and $\mathbf{r}\cdot\mathbf{a} = -3$, where $\mathbf{a}$ is $2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$ cannot be

(a) $\sqrt{5}$ units *(b)* $\sqrt{3}$ units

(c) $2\sqrt{2}$ units *(d)* $\sqrt{2}$ units

Sol. ***(a, c, d)*** Any point of the line can be P

$(1 - k, 1 + k, 10 - k)$

If P lies on $\mathbf{r}\cdot\mathbf{a} = 3$

$\Rightarrow \quad (1 - k)\,2 + (1 + k)\,(-3) + 10 - k = 3$

$\Rightarrow \quad k = 1$

$\therefore$ Point on first plane = $(0, 2, 9)$

Similarly, if P lies on $\mathbf{r}\cdot\mathbf{a} = -3$

$\Rightarrow \quad (1 - k)2 + (1 + k)\,(-3) + 10 - k = -3$

$\Rightarrow \quad k = 2$

$\therefore$ Point on second plane = $(-1, 3, 8)$

Hence, distance between points = $\sqrt{3}$ units

Length of the intercept = $\sqrt{3}$ units

9. If sides of the ΔABC are represented by the lines $\mathbf{r} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \lambda\,(\hat{\mathbf{j}} - \hat{\mathbf{k}})$, $\mathbf{r} = \hat{\mathbf{j}} + \hat{\mathbf{k}} + \mu(\hat{\mathbf{k}} - \hat{\mathbf{i}})$ and $\mathbf{r} = \hat{\mathbf{k}} + \hat{\mathbf{i}} + t\,(\hat{\mathbf{i}} - \hat{\mathbf{j}})$, then (where, $\lambda, \mu, t \in R$)

(a) vertices of triangle are (1, 1, 0), (1, 0, 1) and (0, 1, 1)

(b) area of ΔABC is $\frac{\sqrt{3}}{2}$ sq units

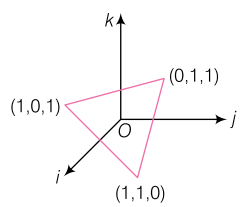
(c) (0, 0, 0) lies in the plane of ΔABC

(d) ΔABC is an equilateral triangle

Sol. ***(a, b, d)*** Clearly, general points on three lines are $(1, \lambda + 1, -\lambda)$, $(-\mu, 1, \mu + 1)$ and $(t + 1, -t, 1)$ respectively.

$\Rightarrow$ Vertices of ΔABC are given by

$$\begin{cases} -\mu = 1 \\ \lambda + 1 = 1 \\ \mu + 1 = -\lambda \end{cases} \quad \begin{cases} -\mu = t + 1 \\ -t = 1 \\ \mu + 1 = 1 \end{cases} \text{ and } \quad \begin{cases} t + 1 = 1 \\ -t = \lambda + 1 \\ -\lambda = 1 \end{cases}$$



$$\begin{cases} \mu = -1, \lambda = 0 \Rightarrow \text{vertex is } (1,\ 1,\ 0) \\ t = -1, \mu = 0 \Rightarrow \text{vertex is } (0,\ 1,\ 1) \\ \lambda = -1, t = 0 \Rightarrow \text{vertex is } (1,\ 0,\ 1) \end{cases}$$

which are vertices of equilateral triangle, whose side is $\sqrt{2}$.

Now, clearly by plotting the vertices on orthogonal triad, we see that (0, 0, 0) does not lie on the plane of ΔABC.

10. For three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ which of the following is always true?

(a) $\forall\ k \in R$, points with position vectors $(\mathbf{a} + \mathbf{b})$, $(\mathbf{a} + 2\mathbf{b})$ and $(\mathbf{a} + k\,\mathbf{b})$ are always collinear

(b) $[(\mathbf{a} - \mathbf{b}) \times (\mathbf{b} - \mathbf{c})] \cdot (\mathbf{c} - \mathbf{a}) = 0$

(c) For some non-zero $\mathbf{x}$, if $\mathbf{a}\cdot\mathbf{x} = 0$, $\mathbf{b}\cdot\mathbf{x} = 0$, $\mathbf{c}\cdot\mathbf{x} = 0$, then $[\mathbf{a}\,\mathbf{b}\,\mathbf{c}] = 0$

(d) $\mathbf{a}\cdot[(\mathbf{b} + \mathbf{c}) \times (\mathbf{a} + \mathbf{b} + \mathbf{c})] = 2[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]$

Sol. ***(a, b, c)***

(a) $[(\mathbf{a} + 2\,\mathbf{b}) - (\mathbf{a} + \mathbf{b})] = \lambda\,[(\mathbf{a} + k\,\mathbf{b}) - (\mathbf{a} + 2\mathbf{b})]$ some λ

$\Rightarrow \mathbf{b} = \lambda[(k - 2)\mathbf{b}]$, which is true.

(b) $[\mathbf{a} - \mathbf{b}\ \ \mathbf{b} - \mathbf{c}\ \ \mathbf{c} - \mathbf{a}] = 0$, which is true.

(c) $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are perpendicular to $\mathbf{x}$ and hence coplanar

$\therefore$ (c) is true.

(d) $\mathbf{a}\cdot[(\mathbf{b} + \mathbf{c}) \times (\mathbf{a} + \mathbf{b} + \mathbf{c})] = \mathbf{a}\cdot[(\mathbf{b} + \mathbf{c}) \times \mathbf{a}] = 0$

$\therefore$ (d) is false.

11. Let equation of three planes are given as $\mathbf{r}\cdot(a\hat{\mathbf{i}}+\hat{\mathbf{j}}+5\hat{\mathbf{k}})=0$, $\mathbf{r}\cdot(2\hat{\mathbf{i}}+2\hat{\mathbf{j}}+a\hat{\mathbf{k}})=0$ and $\mathbf{r}\cdot(5\hat{\mathbf{i}}+6\hat{\mathbf{j}}-\hat{\mathbf{k}})=0$. If these planes intersect at a point other than origin, then b is sum of all possible values of a and c is product of all possible values of a. Then, which of the following is (are) true?

(a) Atleast one value of a is positive

(b) One value of a lies in (1, 2)

(c) Number of rational terms in expansion of $(2^b+3^c)^{10}$ is 6

(d) Sum of an infinite GP, whose first term is c and common ratio is b, is 4

Sol. ***(a, b, c)*** If planes intersect at a point other than origin.

Then, $\begin{vmatrix} a & 1 & 5 \\ 2 & 2 & a \\ 5 & 6 & -1 \end{vmatrix} = 0$

$\Rightarrow \quad 2a^2 - a - 4 = 0 \{^{a_1}_{a_2}$

$\Rightarrow \quad b = a_1 + a_2 = \frac{1}{2}, c = a_1a_2 = -2$

(a) One of a_1 and a_2 is positive and other is negative.

(b) $f(a) = 2a^2 - a - 4$

$f(1) = 2 - 1 - 4 = -3$

$f(2) = 8 - 2 - 4 = 2$

So, one value of a lies in (1, 2).

(c) $(2^{1/2} + 3^{-2})^{10}$ number of rational terms is 6.

(d) Sum of GP $= \frac{c}{1-b} = -\frac{2}{1-\frac{1}{2}} = -4$

12. Assuming the plane $4x - 3y + 7z = 0$ to be horizontal, the equation of the line of greatest slope through the point (2, 1, 1) in the plane $2x + y - 5z = 0$ is/are

(a) $\frac{x-2}{3} = \frac{y-1}{-1} = \frac{z-1}{1}$ *(b)* $\frac{x-2}{-1} = \frac{y-1}{3} = \frac{z-1}{1}$

(c) $\frac{x-2}{-3} = \frac{y-1}{1} = \frac{z-1}{-1}$ *(d)* $\frac{x-2}{1} = \frac{y-1}{-3} = \frac{z-1}{-1}$

Sol. ***(a, c)*** The required line passing through the point P (2, 1, 1) in the plane $2x + y - 5z = 0$ and is having greatest slope, so it must be perpendicular to the line of intersection of the planes

$2x + y - 5z = 0$ ...(i)

and $4x - 3y + 7z = 0$ ...(ii)

Let the direction ratios of the line of intersection of Eqs. (i) and (ii) are a, b and c.

$\Rightarrow \quad 2a + b - 5c = 0$ and $4a - 3b + 7c = 0$

[as normal to both the planes perpendicular to the line of intersection]

$\Rightarrow \quad \frac{a}{4} = \frac{b}{17} = \frac{c}{5}$

Now, let the direction ratio of required line be proportional to l, m, n, then its equation be $\frac{x-2}{l} = \frac{y-1}{m} = \frac{z-1}{n}$.

where, $2l + m - 5n = 0$ and $4l + 17m + 5n = 0$

So, $\frac{l}{3} = \frac{m}{-1} = \frac{n}{1}$

Thus, the required line is $\frac{x-2}{3} = \frac{y-1}{-1} = \frac{z-1}{1}$.

Paragraph (Q. Nos. 13-14)

A plane $\prod$ is determined by three points A (0, 0, 1); B (2, 0, 0) and C (0, 3, 0) and a line L whose vector equation is $\mathbf{r} = \frac{8}{3}\hat{\mathbf{i}} + \hat{\mathbf{k}} + \lambda(2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$ intersects the plane $\prod$ at P.

13. Distance of the point P from the origin is

(a) $\frac{3\sqrt{10}}{2}$ *(b)* $\frac{2\sqrt{10}}{3}$ *(c)* $\sqrt{10}$ *(d)* $2\sqrt{10}$

Sol. ***(b)*** Equation of the plane through (0, 0, 1), (2, 0, 0), (0, 3, 0) is

$\frac{x}{2} + \frac{y}{3} + \frac{z}{1} = 1 \Rightarrow 3x + 2y + 6z = 6$ ...(i)

Equation of L is $\mathbf{r} = \frac{8}{3}\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + \hat{\mathbf{k}} + \lambda(2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$ ...(ii)

Coordinate of any point on Eq. (i) is $\left(2\lambda + \frac{8}{3}\right), -2\lambda, \lambda + 1$.

This lies on $3x + 2y + 6z = 6$.

$3\left(2\lambda + \frac{8}{3}\right) - 4\lambda + 6(\lambda + 1) = 6$

$\Rightarrow \quad 6\lambda + 8 - 4\lambda + 6\lambda + 6 = 6$

$\Rightarrow \quad 8\lambda = -8 \Rightarrow \lambda = -1$

So, P is $\left(\frac{2}{3}, 2, 0\right)$.

Hence, distance of origin from P is $\sqrt{\frac{4}{9} + 4} = \frac{2\sqrt{10}}{3}$

14. A unit vector along the line of intersection of the plane Π and the plane $2x + y - 2z = 5$, is

(a) $\frac{10\hat{\mathbf{i}} - 18\hat{\mathbf{j}} + \hat{\mathbf{k}}}{5\sqrt{17}}$ *(b)* $\frac{10\hat{\mathbf{i}} + 18\hat{\mathbf{j}} + \hat{\mathbf{k}}}{5\sqrt{17}}$

(c) $\frac{10\hat{\mathbf{i}} - 18\hat{\mathbf{j}} - \hat{\mathbf{k}}}{5\sqrt{15}}$ *(d)* $\pm\left(\frac{10\hat{\mathbf{i}} + 18\hat{\mathbf{j}} - \hat{\mathbf{k}}}{5\sqrt{15}}\right)$

Sol. ***(a)*** Vector perpendicular to given plane is $\mathbf{n}_1 = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$

Vector perpendicular to plane

$2x + y - 2z = 5$ is $\mathbf{n}_2 = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$

Vector along the line of intersection of these plane

$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 2 & 6 \\ 2 & 1 & -2 \end{vmatrix} = \hat{\mathbf{i}}(-4-6) - \hat{\mathbf{j}}(-6-12) + \hat{\mathbf{k}}(3-4)$

$= -10\hat{\mathbf{i}} + 18\hat{\mathbf{j}} - \hat{\mathbf{k}}$

Hence, unit vector along the line of intersection is $\pm\frac{10\hat{\mathbf{i}} - 18\hat{\mathbf{j}} + \hat{\mathbf{k}}}{5\sqrt{17}}$.

Paragraph (Q. Nos. 15-16)

Consider, a plane $ax + \sqrt{2}by + cz + d = 0$. Let perpendicular distance of the plane from (0, 0, 0) is $\frac{1}{2}$, where a, b, c and d are in GP.

15. Number of possible planes is

(a) 0 *(b)* 1 *(c)* 2 *(d)* 3

Sol. ***(c)*** The plane will become

$$ax + \sqrt{2}ary + ar^2z + ar^3 = 0$$

$$\Rightarrow \quad x + \sqrt{2}ry + r^2z + r^3 = 0$$

Distance from origin $= \frac{1}{2}$

$$\Rightarrow \quad \frac{1}{2} = \left|\frac{r^3}{\sqrt{1 + 2r^2 + r^4}}\right| \Rightarrow \frac{1}{2} = \left|\frac{r^3}{(r^2 + 1)}\right|$$

$$r = 1, -1$$

16. If the given plane cuts coordinate axes at A, B, C, then volume of tetrahedron $AOBC$ is (where, O is origin)

(a) $\frac{1}{\sqrt{2}}$ *(b)* $\frac{1}{6\sqrt{2}}$

(c) $6\sqrt{2}$ *(d)* $\frac{1}{6}$

Sol. ***(b)*** $r = 1 \Rightarrow a = b = c = d$

Plane is $x + \sqrt{2}y + z + 1 = 0$

$A(-1, 0, 0)$, $B\left(0, -\frac{1}{\sqrt{2}}, 0\right)$ and $C(0, 0, -1)$

$$\therefore \text{ Volume} = \left|\frac{1}{6}\left[\hat{i}\ \frac{\hat{j}}{\sqrt{2}}\ \hat{k}\right]\right| = \frac{1}{6\sqrt{2}}$$

Matching List Type (Only One Option Correct)

This section contains two questions, each having two matching lists. Choices for the correct combination of elements from List I and List II are given as option (a), (b), (c) and (d) out of which one is correct.

17.

List I	List II
A. $\mathbf{a}, \mathbf{b}$ are unit vectors and $(\mathbf{a} + 2\mathbf{b}) \perp (5\mathbf{a} - 4\mathbf{b})$, then $2(\mathbf{a} \cdot \mathbf{b})$ is equal to	1. 0
B. The points (1, 0, 3), (– 1, 3, 4), (1, 2, 1), (k, 2, 5) are coplanar, then k is equal to	2. – 1
C. The vectors (1, 1, m), (1, 1, m + 1) and (1, – 1, m) are coplanar, then number of values of m is	3. 1
D. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b})$ is equal to	4. 2

Codes

	A	B	C	D
(a)	1	2	3	4
(c)	1	3	2	1

	A	B	C	D
(b)	2	4	1	3
(d)	3	2	1	1

Sol. ***(d)*** (A) $\mathbf{a}$ and $\mathbf{b}$ are unit vectors

$$(\mathbf{a} + 2\mathbf{b}) \cdot (5\mathbf{a} - 4\mathbf{b}) = 0$$

$$\Rightarrow \quad 5|\mathbf{a}|^2 + 6\mathbf{a} \cdot \mathbf{b} - 8|\mathbf{b}|^2 = 0$$

$$\Rightarrow \quad 6\mathbf{a} \cdot \mathbf{b} = 3$$

$$\Rightarrow \quad \mathbf{a} \cdot \mathbf{b} = \frac{1}{2}$$

$$\Rightarrow \quad 2(\mathbf{a} \cdot \mathbf{b}) = 1$$

(B) $\mathbf{a} = (1, 0, 3)$, $\mathbf{b} = (-1, 3, 4)$, $\mathbf{c} = (1, 2, 1)$ and $\mathbf{d} = (k, 2, 5)$ are coplanar, if $[\mathbf{dbc}] + [\mathbf{dca}] + [\mathbf{dab}] = [\mathbf{abc}]$

$$\Rightarrow \begin{vmatrix} k & 2 & 5 \\ -1 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} + \begin{vmatrix} k & 2 & 5 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{vmatrix} + \begin{vmatrix} k & 2 & 5 \\ 1 & 0 & 3 \\ -1 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 3 \\ -1 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix}$$

$$\Rightarrow \quad k = -1$$

(C) (1, 1, m), (1, 1, m + 1), (1, – 1, m) are coplanar, if

$$\begin{vmatrix} 1 & 1 & m \\ 1 & 1 & m+1 \\ 1 & -1 & m \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & 0 & m \\ 1 & 0 & m+1 \\ 1 & -2 & m \end{vmatrix} = 0$$

$\Rightarrow 2 = 0$, which is absurd.

$\therefore$ There is no value of m for which the vectors are coplanar.

(D) $$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

$$\mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = (\mathbf{b} \cdot \mathbf{a})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$$

$$\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{c} \cdot \mathbf{b})\mathbf{a} - (\mathbf{c} \cdot \mathbf{a})\mathbf{b}$$

Since, dot product is commutative, so we have the sum = 0

18.

List I	List II
A. If $\lvert\mathbf{a}\rvert = \lvert\mathbf{b}\rvert = \lvert\mathbf{c}\rvert$, angle between each pair of vectors is $\frac{\pi}{3}$ and $\lvert\mathbf{a} + \mathbf{b} + \mathbf{c}\rvert = \sqrt{6}$, then $2\lvert\mathbf{a}\rvert$ is equal to	1. 3
B. If $\mathbf{a}$ is a perpendicular to $\mathbf{b} + \mathbf{c}$, $\mathbf{b}$ is perpendicular to $\mathbf{c} + \mathbf{a}$, $\mathbf{c}$ is perpendicular to $\mathbf{a} + \mathbf{b}$, $\lvert\mathbf{a}\rvert = 2$, $\lvert\mathbf{b}\rvert = 3$ and $\lvert\mathbf{c}\rvert = 6$. Then, $\lvert\mathbf{a} + \mathbf{b} + \mathbf{c}\rvert - 2$ is equal to	2. 2
C. If $\mathbf{a} = 2\hat{i} + 3\hat{j} - \hat{k}$, $\mathbf{b} = -\hat{i} + 2\hat{j} - 4\hat{k}$, $\mathbf{c} = \hat{i} + \hat{j} + \hat{k}$ and $\mathbf{d} = 3\hat{i} + 2\hat{j} + \hat{k}$, then $\frac{1}{7}(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$ is equal to	3. 4
D. If $\lvert\mathbf{a}\rvert = \lvert\mathbf{b}\rvert = \lvert\mathbf{c}\rvert = 2$ and $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} = 2$, then $[\mathbf{abc}] \cos 45°$ is equal to	4. 5

Codes

	A	B	C	D
(a)	1	2	3	4
(c)	1	3	2	1

	A	B	C	D
(b)	2	4	1	3
(d)	3	4	1	2

Sol. ***(b)*** (A) $|\mathbf{a} + \mathbf{b} + \mathbf{c}| = \sqrt{6}$

$$\Rightarrow \mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2 + 2\mathbf{a} \cdot \mathbf{b} + 2\mathbf{b} \cdot \mathbf{c} + 2\mathbf{c} \cdot \mathbf{a} = 6$$

$$\therefore \quad |\mathbf{a}| = 1$$

or $$2|\mathbf{a}| = 2$$

(B) $\mathbf{a}$ is perpendicular to $\mathbf{b} + \mathbf{c}$.

$$\Rightarrow \quad \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} = 0 \qquad \text{...(i)}$$

$\mathbf{b}$ is perpendicular to $\mathbf{a}+\mathbf{c}$.

$\Rightarrow \quad \mathbf{b}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{c}=0$ ...(ii)

$\mathbf{c}$ is perpendicular to $\mathbf{a}+\mathbf{b}$.

$\Rightarrow \quad \mathbf{c}\cdot\mathbf{a}+\mathbf{c}\cdot\mathbf{b}=0$ ...(iii)

From Eqs. (i), (ii) and (iii), we get

$$\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{a}=0$$

$\therefore \quad |\mathbf{a}+\mathbf{b}+\mathbf{c}|=7$

or $\quad |\mathbf{a}+\mathbf{b}+\mathbf{c}|-2=7-2=5$

(C) $(\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d})-(\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d})=21$

(D) We know that, $[\mathbf{a}\times\mathbf{b}\;\mathbf{b}\times\mathbf{c}\;\mathbf{c}\times\mathbf{a}]=[\mathbf{abc}]^2$

and $[\mathbf{abc}]^2=\begin{vmatrix}\mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c}\\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c}\\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{c}\end{vmatrix}=\begin{vmatrix}4 & 2 & 2\\ 2 & 4 & 2\\ 2 & 2 & 4\end{vmatrix}=32$

$\therefore \quad [\mathbf{abc}]=4\sqrt{2}$

$\Rightarrow [\mathbf{abc}]\cos 45^\circ=4\sqrt{2}\times\cos 45^\circ=4\sqrt{2}\times\frac{1}{\sqrt{2}}=4$

Integer Type Questions

19. If the length of shortest distance between the two lines $\frac{1}{2}(x-1)=\frac{1}{4}(y-3)=z+2$ and $3x-y-2z+4=0=2x+y+z+1$ is $\sqrt{\frac{a}{b}}$ where, a and b are coprime, then the value of $a+b-30$ is

Sol. ***(9)*** Any plane through the second line is given by

$$(3x-y-2z+4)+\lambda(2x+y+z+1)=0$$
$$(3+2\lambda)x+(\lambda-1)y+(\lambda-2)z+(4+\lambda)=0$$

If this plane is parallel to first line, then its normal must be at right angles to first line

$$(3+2\lambda)2+(\lambda-1)4+(\lambda-2)=0\Rightarrow\lambda=0$$

$\therefore$ Equation of plane through the second line and parallel to first line is $3x-y-2z+4=0$.

Shortest distance (SD) = Perpendicular distance of a point $(1, 3, -2)$ on the first line to the plane $3x-y-2z+4=0$

$$\Rightarrow \text{SD}=\frac{|3-3+4+4|}{\sqrt{3^2+1^2+2^2}}=\frac{8}{\sqrt{14}}=\sqrt{\frac{64}{14}}=\sqrt{\frac{32}{7}}=\sqrt{\frac{a}{b}}$$

$$a=32 \quad \text{and} \quad b=7$$

Now, $\quad (a+b)=39$

$\Rightarrow \quad a+b-30=9$

20. Let the image of straight line $\frac{x-1}{3}=\frac{y-3}{5}=\frac{z-4}{2}$ with plane $2x-y+z+3=0$ is L and one plane $7x+by+cz+d=0$ is such that it contains the straight line L and is perpendicular to the plane $2x-y+z+3=0$, then $b+c+d-26$ is equal to

Sol. ***(4)*** Image of $P(1, 3, 4)$ of the line with respect to the plane is $Q(-3, 5, 2)$ and point of intersection of plane and line is $S(-5, -7, 0)$.

Equation of QS is $L: \frac{x+3}{1}=\frac{y-5}{6}=\frac{z-2}{1}$

Equation of planes containing line L are

$$a(x+3)+b(y-5)+c(z-2)=0 \quad \text{...(i)}$$

and $\quad a+6b+c=0$ ...(ii)

This plane is perpendicular to $2x-y+z+3=0$

$\therefore \quad 2a-b+c=0$

On solving, $\frac{a}{7}=\frac{b}{1}=\frac{c}{-13}$

$\Rightarrow \quad 7(x+3)+1(y-5)-13(z-2)=0$

$\Rightarrow \quad 7x+y-13z+42=0$

$\Rightarrow \quad b+c+d=1-13+42=30=26+4$

$\Rightarrow \quad b+c+d-26=4$

21. Let $\mathbf{a}=(2+\sin\theta)\hat{\mathbf{i}}+\cos\theta\hat{\mathbf{j}}+\sin 2\theta\hat{\mathbf{k}}$

$$\mathbf{b}=\sin\left(\theta+\frac{2\pi}{3}\right)\hat{\mathbf{i}}+\cos\left(\theta+\frac{2\pi}{3}\right)\hat{\mathbf{j}}+\sin\left(2\theta+\frac{4\pi}{3}\right)\hat{\mathbf{k}}$$

and $\mathbf{c}=\sin\left(\theta-\frac{2\pi}{3}\right)\hat{\mathbf{i}}+\cos\left(\theta-\frac{2\pi}{3}\right)\hat{\mathbf{j}}+\sin\left(2\theta-\frac{4\pi}{3}\right)\hat{\mathbf{k}}$

be three vectors, where $\theta\in\left(0,\frac{\pi}{2}\right)$. If the volume of tetrahedron, whose coterminus edges are represented by the vector $2\mathbf{b}\times\mathbf{c}$, $3\mathbf{c}\times\mathbf{a}$ and $\mathbf{a}\times 4\mathbf{b}$ is μ, then the value of $\mu-3$ is

Sol. ***(9)*** $[\mathbf{abc}]=\begin{vmatrix}2+\sin\theta & \cos\theta & \sin 2\theta\\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{4\pi}{3}\right)\\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right)\end{vmatrix}$

WB JEE
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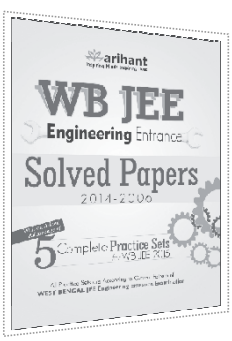
Solved Paper (2014-2006)



5 Complete Practice Sets for WB JEE 2015

On applying $R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} 2+\sin\theta+2\sin\theta\cos\frac{2\pi}{3} & \cos\theta+2\cos\theta\cos\frac{2\pi}{3} & \sin2\theta+2\sin2\theta\cos\frac{4\pi}{3} \\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{4\pi}{3}\right) \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 0 & 0 \\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(2\theta+\frac{4\pi}{3}\right) \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(2\theta-\frac{4\pi}{3}\right) \end{vmatrix}$$

$$= 2\left[\cos\left(\theta+\frac{2\pi}{3}\right)\sin\left(2\theta-\frac{4\pi}{3}\right)-\sin\left(2\theta+\frac{4\pi}{3}\right)\cos\left(\theta-\frac{2\pi}{3}\right)\right]$$

$$= \sin\left(3\theta-\frac{2\pi}{3}\right)-\sin(2\pi-\theta)-\sin\left(3\theta+\frac{2\pi}{3}\right)-\sin(\theta+2\pi)$$

$$= -\left(2\cos3\theta\sin\frac{2\pi}{3}\right) = -\sqrt{3}\cos3\theta$$

$$[\mathbf{abc}]_{max} = \sqrt{3}$$

$$\Rightarrow \quad 3\theta = \pi$$

$$\Rightarrow \quad \theta = \frac{\pi}{3}$$

$$\therefore \text{ Required volume} = \frac{1}{6}[2\,\mathbf{b}\times\mathbf{c}\;3\,\mathbf{c}\times\mathbf{a}\;\mathbf{a}\times4\mathbf{b}]$$

$$= \frac{1}{6}\times24[\mathbf{b}\times\mathbf{c}\;\mathbf{c}\times\mathbf{a}\;\mathbf{a}\times\mathbf{b}] = 4[\mathbf{a}\;\mathbf{b}\;\mathbf{c}]^2$$

Maximum value = $4 \times 3 = 12$ i.e. $\mu = 12$

$$\Rightarrow \quad \mu - 3 = 9$$

22. Suppose $\mathbf{b}$ and $\mathbf{c}$ are non-collinear vectors. If $\mathbf{a}$ is any vector such that $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = 4$ and $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (x^2+2x+6)\mathbf{b} + (\sin y)\mathbf{c}$, then the value of $(x+1)^2$ is

Sol. ***(0)*** $\mathbf{a}\cdot\mathbf{c} = x^2+2x+6$ and $-\mathbf{a}\cdot\mathbf{b} = \sin y$

$$x^2+2x+6-\sin y = 4 \Rightarrow (x+1)^2+1 = \sin y$$

$$\Rightarrow \quad (x+1)^2 = \sin y - 1$$

$$\because \quad (x+1)^2 \geq 0$$

$$\Rightarrow \quad \sin y - 1 \geq 0$$

$$\Rightarrow \quad \sin y \geq 1$$

$$\Rightarrow \quad \sin y = 1$$

$$\Rightarrow \quad (x+1)^2 = 0$$

23. Let $\mathbf{a} = \lambda\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} + 2\lambda\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ and $\mathbf{c} = 2\hat{\mathbf{i}} - \lambda\hat{\mathbf{j}} + \hat{\mathbf{k}}$. If the value of λ for which $\{(\mathbf{a}\times\mathbf{b})\times(\mathbf{b}\times\mathbf{c})\}\times(\mathbf{c}\times\mathbf{a}) = 0$ is $\frac{p}{q}$, then the value of $p+q$ is

Sol. ***(5)*** Given that, $\{(\mathbf{a}\times\mathbf{b})\times(\mathbf{b}\times\mathbf{c})\}\times(\mathbf{c}\times\mathbf{a}) = 0$

$$\Rightarrow \quad \{[\mathbf{abc}]\,\mathbf{b} - [\mathbf{bbc}]\,\mathbf{c}\}\times(\mathbf{c}\times\mathbf{a}) = 0$$

$$\Rightarrow \quad [\mathbf{abc}]\{\mathbf{b}\times(\mathbf{c}\times\mathbf{a})\} = 0$$

$$\Rightarrow \quad [\mathbf{abc}] = 0$$

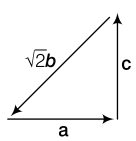
$$\Rightarrow \quad \begin{vmatrix} \lambda & 2 & -3 \\ 1 & 2\lambda & -2 \\ 2 & -\lambda & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda = \frac{2}{3} = \frac{p}{q}$$

$$\therefore \quad p+q = 3+2 = 5$$

24. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are unit vectors such that $\mathbf{a}+\sqrt{2}\mathbf{b}+\mathbf{c} = 0$, then the value of $[|2\mathbf{a}+3\mathbf{b}-2\mathbf{c}|]$ (where, $[\cdot]$ denotes greatest integer function) will be

Sol. ***(4)*** $\mathbf{a}\cdot\mathbf{c} = 0; \mathbf{b}\cdot\mathbf{a} = \frac{-1}{\sqrt{2}}; \mathbf{c}\cdot\mathbf{b} = \frac{-1}{\sqrt{2}}$



$$|2\mathbf{a}+3\mathbf{b}-2\mathbf{c}|$$

$$= \sqrt{4a^2+9b^2+4c^2+12\mathbf{a}\cdot\mathbf{b}-12\mathbf{b}\cdot\mathbf{c}-8\mathbf{c}\cdot\mathbf{a}} = \sqrt{17}$$

$$\therefore |2a+3b-2c| = 4$$

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TEST DRIVE

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SCHOOL PRACTICE RIDE 1

General Instructions

1. *All questions are compulsory.*
2. *The question paper consists of* 26 *questions divided into three sections A, B and C. Section A comprises of* 06 *questions of one mark each, Section B comprises of* 13 *questions of four marks each and Section C comprises of* 07 *questions of six marks each.*
3. *All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.*
4. *There is no overall choice. However, internal choice has been provided in* 04 *questions of four marks each and* 02 *questions of six marks each. You have to attempt only one of the alternative in all such questions.*
5. *Use of calculator is not permitted. You may ask for logarithmic table, if required.*

Section A

1. If $\sin\theta + \text{cosec}\theta = 2$, then find $\sin^2\theta + \text{cosec}\ \theta$.

2. If $n(A) = 4$, $n(B) = 5$, $n(U) = 7$ and $n(A \cap B) = 2$, then find the value of $n(A \cup B)'$.

3. Find x and y, if $(x + 6, y - 2) = (0, 6)$.

4. Compute $\dfrac{20!}{18!(20-18)!}$.

5. If a parallelopiped is formed by planes drawn through the points (5, 8, 10) and (3, 6, 8) parallel to the coordinates planes, then find the length of diagonal of the parallelopiped.

6. Evaluate $\lim\limits_{x \to 2}(x^4 + 3x^2 + 2)$.

Section B

7. The angles of a triangle are in AP. The number of degree in the least is to the number of radians in the greatest is as $60 : \pi$. Find the angles in degree.

or

Find the distance from the eye at which a coin of diameter 2 cm should be held so as just to conceal the full moon, whose angular diameter is 31′.

8. Prove that $A - (B \cup C) = (A - B) \cap (A - C)$.

or If $U = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$,
$A = \{2, 4, 7\}$, $B = \{3, 5, 7, 9, 11\}$
and $C = \{7, 8, 9, 10, 11\}$, then compute

(i) $(A \cap U) \cap (B \cup C)$. (ii) $C - B$.

(iii) $B - C$. (iv) $(B - C)'$.

9. If $y = f(x) = \dfrac{1-x}{1+x}$, then show that $x = f(y)$.

10. Draw the graph of the function $f : R \to R$ such that $f(x) = |x - 2|$.

11. In any ΔABC, prove that $\dfrac{\sin(B-C)}{\sin(B+C)} = \dfrac{b^2 - c^2}{a^2}$.

or

Solve the equation $\sin\theta + \sin 3\theta + \sin 5\theta = 0$.

12. Prove by mathematical induction that the sum of first n odd natural numbers is n^2.

13. A committee of 3 persons is to be constituted from a group of 2 men and 3 women. In how many ways can this be done? How many of these committee would consists of 1 man and 2 women? Identify the value being highlighted in this question. [VBQ]

14. Write down the negation of each of the following:

(i) $3 + 5 = 8$ and $2 + 4 > 7$

(ii) If he works hard, he will pass the examination

(iii) If it rains, he will not go for a walk

(iv) Preeti is tall and therefore she is slim

15. The probability that a student will pass the final examination in both English and Hindi is 0.5 and the probability of passing neither is 0.1. If the probability of passing the English examination is 0.75, then what is the probability of passing the Hindi examination?

16. Solve $2x^2 - 2\sqrt{3}x + \frac{21}{8} = 0$. *or* Solve $x^2 - 5ix - 6 = 0$.

17. Two unbiased dice are thrown. Find the probability that neither a doublet nor a total of 10 will appear.

18. The sum of first three terms of a GP is $\frac{13}{12}$ and their product is -1. Find the terms.

19. Evaluate $\lim_{x \to 2} \frac{x^3 - 7x^2 + 14x - 8}{x^2 + 2x - 8}$.

Section C

20. Two consecutive sides of parallelogram are $4x + 5y = 0$ and $7x + 2y = 0$. If the equation of one diagonal is $11x + 7y = 9$, then find the equation of other diagonal.

21. Find the equation of the circle which is circumscribed about the triangle, whose vertices are $(-2, 3)$, $(5, 2)$ and $(6, -1)$.

22. Solve the system of inequalities graphically.

$x + y \le 4$, $3x + y \ge 3$

$x + 4y \ge 4$, $x \le 3, y \le 2$

or

Solve graphically the following system of inequalities:

$x + 2y \le 3$, $3x + 4y \ge 12$

$x \ge 0$, $y \ge 1$

23. The AM and SD of 100 items was recorded as 40 and 5.1, respectively. Later on, it was discovered that one observation 40 was wrongly copied down as 50. Find the correct SD.

24. Find the term independent of x in the expansion of $(1 + x + 2x^3)\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$.

25. Find four numbers in GP, whose sum is 85 and product is 4096.

or

Find the sum of series $\frac{3}{5} + \frac{4}{5^2} + \frac{3}{5^3} + \frac{4}{5^4} + \ldots + 2n$ terms.

26. A college awarded 38 medals in football, 15 in basketball and 20 medals in cricket. If these medals went to a total of 58 men and only three men got medals in all the three sports, then how many received medals in exactly two of the three sports.

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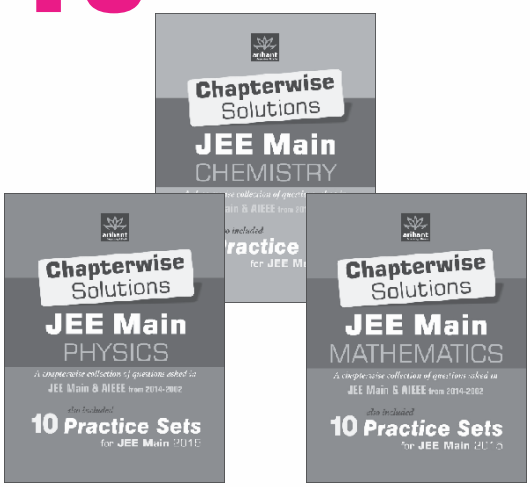

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TEST DRIVE

The Simulator test series towards School Practice and Board Exam

BOARD EXAM RIDE 1

General Instructions

(i) *All questions are compulsory.*

(ii) *The question paper consists of* 26 *Questions divided into three sections A, B and C. Section A comprises of* 06 *questions of one mark each, Section B comprises of* 13 *questions of four marks each and Section C comprises of* 07 *questions of six marks each.*

(iii) *All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.*

(iv) *There is no overall choice. However, internal choice has been provided in* 04 *questions of four marks each and* 02 *questions of six marks each. You have to attempt only one of the alternatives in all such questions.*

(v) *Use of calculators is not permitted. You may ask for logarithmic tables, if required.*

Section A

1. Find the principal value of $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)+\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$.

2. If $\begin{pmatrix} a+b & 2 \\ 5 & b \end{pmatrix}=\begin{pmatrix} 2 & 2 \\ 5 & 9 \end{pmatrix}$, then find the value of a.

3. Find the projection of vector $\hat{i}+\hat{j}$ over $\hat{k}+\hat{i}$.

4. Using the third row of the determinant $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 9 \\ 1 & 1 & 2 \end{vmatrix}$, write its value.

5. Find the number of parameters in $y = a\sin x + b\cos(x + c)$.

6. Show that the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x+1}{1}=\frac{y+2}{2}=\frac{z+3}{-2}$ are perpendicular.

Section B

7. Let T be the set of all triangles in a plane. Let us define a relation $R=\{(T_1,T_2):T_1 \text{is similar to } T_2; T_1, T_2 \in T\}$

Show that R is an equivalence relation.

or

Consider $f: R_t \to [4,\infty)$ given by $f(x)=x^2+4$.

Show that f is invertible with the inverse (f^{-1}) of f given by $f^{-1}(y)=\sqrt{y-4}$, where R_t is set of all non-negative real numbers.

8. Prove that

$$\tan\left\{\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)+\frac{1}{2}\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\right\}=\frac{x+y}{1-xy}.$$

or

Find the value of $\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right)$.

9. If a, b and c are in AP, then show that the determinant $\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}=0$.

10. Find the values of a and b such that the function f defined by $f(x)=\begin{cases} \frac{x-4}{|x-4|}+a, & \text{if } x<4 \\ a+b, & \text{if } x=4 \\ \frac{x-4}{|x-4|}+b, & \text{if } x>4 \end{cases}$ is continuous function at $x=4$

11. Verify Rolle's theorem for the function

$$f(x) = \sin x + \cos x, \text{ where } x \in \left[0, \frac{\pi}{2}\right].$$

12. If $\cos y = x\cos(a + y), \cos a \neq 1$, then prove that $\frac{dy}{dx} = \frac{\cos^2(a + y)}{\sin a}$.

13. Solve the following differential equation

$$\left(1 + e^{\frac{x}{y}}\right) dx + e^{\frac{x}{y}}\left(1 - \frac{x}{y}\right) dy = 0$$

or

Solve the differential equation $\frac{dy}{dx} + 2y \tan x = \sin x$, given that $y = 0$, if $x = \frac{\pi}{3}$.

14. Evaluate $\int_{-1}^{\frac{3}{2}} |x \sin(\pi x)|\, dx$.

15. A company has two plants for manufacturing scooters. Plant I manufactures 70% of the scooters and plant II manufactures 30% of the scooters. At plant I, 30% of the scooters are rated of standard quality and at plant II, 90% of the scooters are rated of standard quality. A scooter is chosen at random and is found to be of standard quality. Find the probability that it has come from plant II. Write the features of a place, where a manufacturing plant can be started.

16. If $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$ and $|\mathbf{a}| = 3, |\mathbf{b}| = 5$ and $|\mathbf{c}| = 7$, then show that the angle between $\mathbf{a}$ and $\mathbf{b}$ is 60°.

17. Evaluate $\int \frac{xe^{2x}}{(1 + 2x)^2} dx$.

or

Evaluate $\int \frac{dx}{\sin(x - a)\cos(x - b)}$.

18. Find the distance from the point (3, 4, 5) to the point, where the line $\frac{x - 3}{1} = \frac{y - 4}{2} = \frac{z - 5}{2}$ meets the plane $x + y + z = 2$.

19. Find the equation of tangent to the curves $x = 1 + \cos\theta$ and $y = \theta - \sin\theta$ at $\theta = \frac{\pi}{2}$.

Section C

20. Using the method of integration, find the area of the region bounded by the following lines $5x - 2y - 10 = 0, x + y - 9 = 0$ and $2x - 5y - 4 = 0$.

21. Solve the following system of equations by matrix method when $x \neq 0, y \neq 0$ and $z \neq 0$

$$\frac{2}{x} - \frac{3}{y} + \frac{3}{z} = 10$$
$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 10$$
$$\frac{3}{x} - \frac{1}{y} + \frac{2}{z} = 13$$

22. Suppose, 6% of men and 0.25% of women have gray hair. A gray haired person is selected at random. What is the probability of this person being male? Assume that there are equal number of males and females.

or

An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probabilities of an accident involving a scooter, a car and a truck are 0.01, 0.03 and 0.15, respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver?

23. Find the maximum and minimum values of $x + \sin 2x$ in the interval $[0, 2\pi]$.

or

A given quantity of metal is to be cast into a solid half circular cylinder with rectangular base and semi-circular ends. Show that in order that the total surface area may be minimum the ratio of the length of the cylinder to the diameter of its circular ends is $\pi : (\pi + 2)$.

24. David wants to invest atmost ₹ 12000 in bonds *A* and *B*. According to the rule, he has to invest atleast ₹ 2000 in bond *A* and atleast ₹ 4000 in bond *B*. If the rates of interest on bond *A* and *B* respectively, are 8% and 10% per annum. Formulate the problem as linear programming problem and solve it graphically for maximum interest. Also, determine the maximum interest received in a year.

25. Find the equation of plane passing through three points (1, 0, 0), (2, 1, 9) and (–1, 1, 1).

26. Evaluate $\int \frac{1}{\cos^4 x + \sin^4 x} dx$.-

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for Class XII

TEST DRIVE

The Simulator test series towards School Practice and Board Exam

BOARD EXAM RIDE 2

General Instructions

(i) All questions are compulsory.

(ii) *The question paper consists of* 26 *questions divided into three sections A, B and C. Section A comprises of* 06 *questions of one mark each, Section B comprises of* 13 *questions of four marks each and Section C comprises of* 07 *questions of six marks each.*

(iii) *All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.*

(iv) *There is no overall choice. However, internal choice has been provided in* 04 *questions of four marks each and* 02 *questions of six marks each. You have to attempt only one of the alternatives in all such questions.*

(v) *Use of calculators is not permitted. You may ask for logarithmic tables, if required.*

Section A

1. Write the principal value of $\sin^{-1}\left(\sin\frac{3\pi}{5}\right)$.

2. For what value of k, $\begin{vmatrix} 2k & -1 \\ -8 & 3 \end{vmatrix}$ is a singular matrix?

3. Find the angle between the vectors $\mathbf{a} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$.

4. Find whether $y = \frac{a}{x} + b$ is a solution of $\frac{d^2y}{dx^2} + \frac{2}{x}\left(\frac{dy}{dx}\right) = 0$.

5. If $x\begin{bmatrix}2\\3\end{bmatrix} + y\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}10\\5\end{bmatrix}$, then find x and y.

6. Find the distance of the point (a, b, c) from Y-axis.

Section B

7. If $f : R \to R$ is a function defined by $f(x) = 2x^3 - 5$, show that the function f is a bijective function.

8. Find the intervals in which the function $f(x) = 2x^3 - 9x^2 + 12x + 15$ is
(i) increasing (ii) decreasing.

or

Prove that the curves $x = y^2$ and $xy = k$ cut at right angles, if $8k^2 = 1$.

9. Using elementary transformations, find the inverse of $\begin{bmatrix}4 & 5\\3 & 4\end{bmatrix}$.

or

Using properties of determinants, show that $\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz(x-y)(y-z)(z-x)$.

10. Evaluate $\int (\sqrt{\tan x} + \sqrt{\cot x})\,dx$.

or

Evaluate $\int e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx$.

11. If the function $f(x)$ given by $f(x) = \begin{cases} 3ax + b, & \text{if } x > 1 \\ 11, & \text{if } x = 1 \\ 5ax - 2b, & \text{if } x < 1 \end{cases}$ is continuous at $x = 1$, then find the values of a and b.

or

Find the derivative of $\tan^{-1}\left[\frac{\sqrt{1+x^2}-1}{x}\right]$ with respect to $\tan^{-1}\left[\frac{2x}{1+x^2}\right]$.

12. By computing shortest distance, determine whether the following pair of lines intersect or not:

$$\mathbf{r} = (4\hat{\mathbf{i}} + 5\hat{\mathbf{j}}) + \lambda(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}})$$

and $\mathbf{r} = (\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) + \mu(2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 5\hat{\mathbf{k}})$

13. If the sum of two unit vectors is a unit vector, then show that the magnitude of their difference is $\sqrt{3}$.

14. Of the students in a college, it is known that 60% reside in hostel and 40% are day scholars (not residing in the hostel). Previous year results report that 30% of all students who reside in hostel attain *A* grade and 20% of day scholars attain *A* grade in their annual examination. At the end of the year, one student is chosen at rondom from the college and he has an *A* grade. What is the probability that the student is a hostler? Grading system is better than numerical marking, explain.

15. Using properties of definite integrals, evaluate $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$.

16. Show that $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3$

$$= 2\left(\tan^{-1}1 + \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}\right).$$

17. Evaluate $\int x(\log x)^2\, dx$.

18. If $y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$, then show that

$$(1-x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0.$$

19. Find the equations of the normal to the curve $y = x^3 + 2x + 6$, which are parallel to the line $x + 14y + 4 = 0$.

Section C

20. An aeroplane can carry a maximum of 200 passengers. A profit of ₹ 1000 is made on each executive class ticket and a profit of ₹ 600 is made on each economy class ticket. The airline reserves atleast 20 seats for executive class. However, atleast 4 times as many passengers prefer to travel by economy class than by executive class. Determine how many tickets of each type must be sold in order to maximize profit for the airline? What is the maximum profit? Form a linear programming problem and solve graphically.

21. Show that $\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & b+a & c \end{vmatrix}$

$$= (a+b+c)(a^2+b^2+c^2).$$

22. Suppose a girl throws a die. If she gets a 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4, she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head, then what is the probability that she threw 1, 2, 3 or 4 with the die?

or

A man is known to speak the truth in 3 out of 4 times. He throws a die and reports that it is a six, find the probability that it is actually a six.

23. Find the distance of the point $(-1, -5, -10)$ from the point of intersection of the line $\mathbf{r} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}} + \lambda(3\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$ and the plane $\mathbf{r}\cdot(\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}) = 5$.

24. Show that the semi-vertical angle of right circular cone of given surface area and maximum value is $\sin^{-1}\left(\frac{1}{3}\right)$.

or

An Apache helicopter of enemy is flying along the curve given by $x^2 + 7$. A soldier, placed at (3, 7) wants to shoot down the helicopter when it is nearest to him. Find the nearest distance between the soldier and helicopter.

25. Solve the given differential equation $x^2\frac{dy}{dx} = y^2 + 2xy$, given that $y = 1$ when $x = 1$.

26. Find the area of the region

$\{(x, y) : 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\}$.

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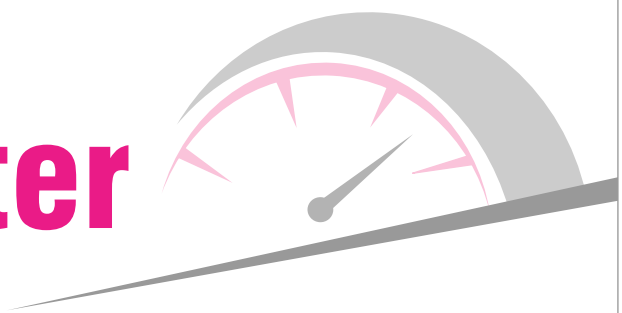
Quizzer (No.3)



1. Let $(5+2\sqrt{6})^n = p + f$, where $n \in N$, $p \in N$ and $0 < f < 1$, then the value of $f^2 - f + pf - p$ is

(a) a natural number (b) a negative number
(c) a prime number (d) an irrational number

2. Angular hexagon $ABCDEF$ is inscribed in a circle of radius 1. A point P is taken on the circle of radius 2 having same centre, then $PA^2 + PB^2 + PC^2 + PD^2 + PE^2 + PF^2$ is equal to

(a) 18 (b) 12
(c) 24 (d) 30

3. Let a denotes the number of non-negative values of p for which the equation $p \cdot 2^x + 2^{-x} = 5$ possess a unique solution, if $\frac{1}{a}, \frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \ldots, \frac{1}{\alpha_{20}}, \frac{1}{6}$ are in AP and $1, \beta_1, \beta_2, \beta_3, \ldots, \beta_{20}, 6$ are in AP, then the value of $\alpha_{18}\beta_3$ is

(a) 10 (b) 11
(c) 9 (d) 12

4. Let α, β be the roots of $x^2 - x - p = 0$ and γ, δ be the roots of $x^2 - 4x + q = 0$. If α, β, γ and δ are in GP, then p^q is equal to

(a) $(3)^{30}$ (b) $\frac{1}{2^{32}}$
(c) $(-2)^{32}$ (d) None of these

5. If $x^2 + x + 1 = 0$, then the numerical value of $\left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)^2 + \left(x^3 + \frac{1}{x^3}\right)^2 + \ldots \left(x^{27} + \frac{1}{x^{27}}\right)^2$ is

(a) 54 (b) 36
(c) 27 (d) 18

6. 12 guest at a dinner party are to be seated along a circular table. Supposing that the master and mistress of the house have fixed seats opposite one another and that there are two specified guest who must always, be placed next to one another, the number of ways in which the company can be placed is

(a) (20) 10! (b) 22 (10)!
(c) 44 (10!) (d) None of these

7. Let $ABCD$ be a square with each side of length 2 units. C_2 is the circle through vertices A, B, C, D and C_1 is the circle touching all sides of the square $ABCD$. L is a line through A. A circle touches the line L and the circle C_1 externally such that both the circles are on the same side of the line, then the locus of the centere of the circle is

(a) ellipse (b) hyperbola
(c) parabola (d) parts of straight line

8. From a point $P(a, b, c)$ perpendiculars PM and PN are drawn to zx and xy-planes, respectively O is origin. An equation of the plane OMN is

(a) $\frac{x}{a} - \frac{y}{b} - \frac{z}{c} = 0$ (b) $\frac{x}{a} - \frac{y}{b} + \frac{z}{c} = 0$
(c) $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0$ (d) $\frac{x}{a} + \frac{y}{b} - \frac{z}{c} = 0$

9. If $f(x) = \frac{4^x}{4^x + 2}$, then the value of $f\left(\frac{1}{1997}\right) + f\left(\frac{2}{1997}\right) + \ldots + f\left(\frac{1996}{1997}\right)$ is

(a) 1000 (b) 998
(c) 999 (d) 989

10. If $I = \int \frac{x^{5/6} - x^{-7/6}}{x^{1/3}(x^2 + x + 1)^{1/2} - x^{1/2}(x^2 + x + 1)^{1/3}}\, dx$ then I is to equal

(a) $2x \log\left(x + \frac{1}{x} + 1\right)^{1/6} - \tan^{-1}\left(\frac{x + \frac{1}{x}}{\sqrt{3}}\right) + C$

(b) $\tan^{-1}\left(\frac{x + \frac{1}{x}}{2}\right) + \log\left(x + \frac{1}{x} + 1\right)^{-3/2} + C$

(c) $\log\left(x + \frac{1}{x} + 7\right)^{1/6} - \tan^{-1}\left(\frac{x^2 - 1}{x}\right)^{-3/2} + C$

(d) None of the above

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Quizzer (No. 3)

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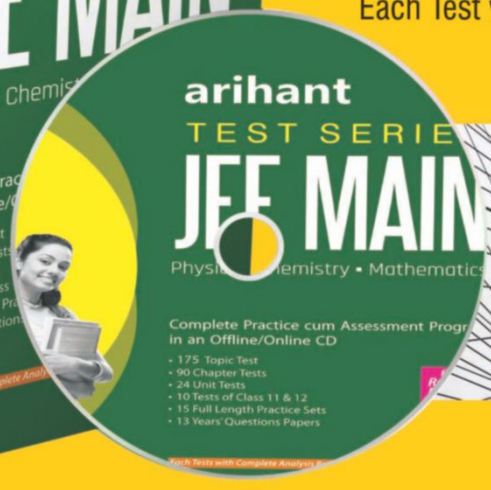
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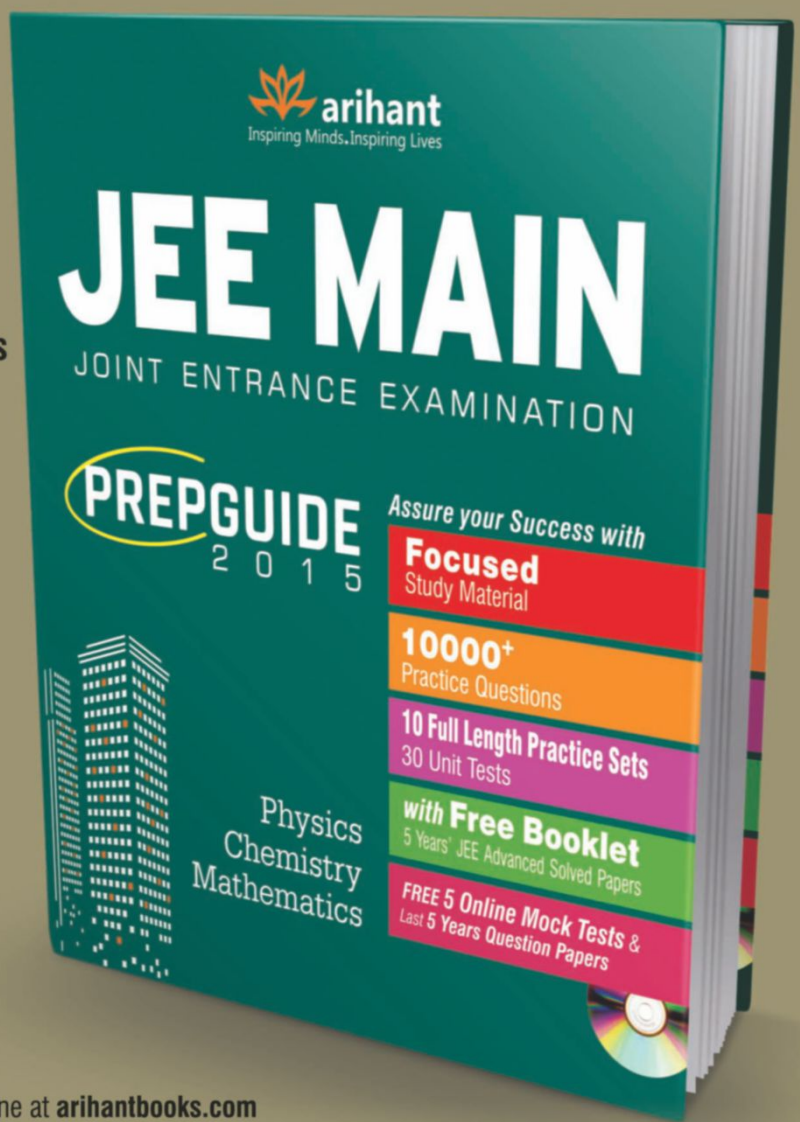

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